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**ANALYTICAL MODEL FOR TILTING PROPROTOR AIRCRAFT  
DYNAMICS, INCLUDING BLADE TORSION AND COUPLED  
BENDING MODES, AND CONVERSION MODE OPERATION**

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ANALYTICAL MODEL FOR TILTING PROPROTOR AIRCRAFT  
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BENDING MODES, AND CONVERSION MODE OPERATION

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SUMMARY

An analytical model is developed for proprotor aircraft dynamics. The rotor model includes coupled flap-lag bending modes, and blade torsion degrees of freedom. The rotor aerodynamic model is generally valid for high and low inflow, and for axial and nonaxial flight. For the rotor support, a cantilever wing is considered; incorporation of a more general support with this rotor model will be a straight-forward matter.

INTRODUCTION

This report presents the development of an analytical model for tilting proprotor aircraft dynamics. The emphasis in this model is on the rotor. The rotor support for the present is limited to a cantilever wing, but the incorporation of a more general support model with this rotor model will be a straight-forward matter.

The rotor motion is represented by: coupled flap and lag bending modes; rigid pitch (control system flexibility) and blade elastic torsion deflection; gimbal tilt and rotor speed perturbation degrees of freedom (optional). The six components of shaft linear and angular motion are included, and rotor blade pitch control. The rotor aerodynamic model is

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generally valid for high and low inflow, and for axial and nonaxial flight. The effects of compressibility and static stall may be included, but reverse flow and unsteady wake aerodynamic interference effects are neglected. Three components of aerodynamic gust are included as external excitation. The rotor model includes gimbal undersling, torque offset, precone, droop, sweep, and feathering axis offset (for the case with blade bending flexibility inboard of the pitch bearing). Center of gravity, aerodynamic center, and tension center offsets are included; but the elastic axis is assumed to be a straight line, and only offset from the pitch axis by the droop and sweep rotations. For the equations of motion in the nonrotating frame it is assumed the rotor has three or more blades. The equations of motion are derived for the rotor degrees of freedom, along with the forces and moments acting on the hub.

This rotor model may be coupled with any support model. The present derivation is restricted to a cantilever wing support (fig. 1). The wing motion is represented by three degrees of freedom: wing vertical bending, wing chordwise bending, and wing torsion. Wing aerodynamic forces are included, and a wing trailing edge flap in the controls.

The differential equations of motion for the proprotor and support system are presented in matrix form, for three cases: axial flow, which is a constant coefficient system; nonaxial flow, which is properly a periodic coefficient system; and a constant coefficient approximation for the nonaxial flow equations, using the mean of the coefficients in the nonrotating frame. The axial flow case is applicable to the proprotor aircraft in airplane mode cruise and in helicopter mode hover flight. The nonaxial flow case is applicable to helicopter mode forward flight, and to conversion mode flight of the proprotor aircraft.

Solutions and results for proprotor dynamics from these equations are not presented in this report, but are left to a later work.

The body of this report is composed of the following sections:

Bending/Torsion of Highly Twisted Beam

Equations of Motion for a Rotating Blade

Aerodynamics

Rotor Trim

Blade Bending and Torsion Modes

Support Equations of Motion: Cantilever Wing

Equations of Motion

## BENDING/TORSION OF HIGHLY TWISTED BEAM

This section presents an engineering beam theory model for the coupled flap/lag bending and torsion of a rotor blade, with large pitch and twist. A high aspect ratio (of the structural elements) is assumed so the beam model is applicable. The object is to relate the bending moments at the section, and the torsion moment, to the blade deflection and elastic torsion at that section. The analysis follows the work of references 1-3.

The basic assumptions are i) an elastic axis exists, and the undeformed elastic axis is a straight line, and ii) the blade has a high aspect ratio (of the structural elements), so engineering beam theory applies. Figure 2 shows the geometry of the undeformed blade. The span variable is  $r$ , measured from the center of rotation along the straight elastic axis. The section coordinates are  $x$  and  $z$ , the principle axes of the section, with origin at the elastic axis. So by definition

$$\int_{\text{section}} xz \, dA = 0$$

Really the integral is over the tension carrying elements, i.e. modulus weighted,  $\int xzE \, dA = 0$ ; so  $x$  and  $z$  are modulus principle axes.

This remark holds for all the section integrals in this section. The tension center (modulus weighted centroid) is  $x_c$  aft of the elastic axis, and on the  $x$  axis, i.e.

$$\begin{aligned}\int x \, dA &= x_c A \\ \int z \, dA &= 0\end{aligned}$$

Again, these are modulus weighted integrals. If  $E$  is uniform over the section, then  $x_c$  is the area centroid; and if the section mass distribution is the same as the  $E$  distribution, then  $x_c$  equals the section center of gravity location.

The angle of the major principle axis (the  $x$  axis) with respect to the hub plane is  $\Theta$ . The existence of the elastic axis means that elastic twist about the EA occurs without bending; we may, and shall, include the elastic torsion deflection in  $\Theta$ . The blade feathering axis (FA) is at

$r_{FA}$ ; the blade pitch is described by root pitch  $\theta^0$  (rigid pitch about the FA, including that due to elastic distortion of the control system), built in twist  $\theta_{tw}$ , and elastic torsion about the EA  $\theta_e$ :

$$\theta = \theta^0 + \theta_{tw} + \theta_e$$

$\theta^0(\psi)$  = root pitch; command collective and cyclic and control system flexibility; rigid pitch about the FA; equals  $\theta$  at  $r_{FA}$

$\theta_{tw}(r)$  = built in twist,  $\theta_{tw}(r_{FA}) = 0$

$\theta_e(r, \psi)$  = elastic torsion,  $\theta_e(r_{FA}, \psi) = 0$

There is stress in the blade due to  $\theta_e$  only. It is assumed that  $\theta_e$  is small, but  $\theta^0$  and  $\theta_{tw}$  are allowed to be large.

The unit vectors in the hub plane (HP) axis system (rotating) are  $\vec{i}_e, \vec{j}_e, \vec{k}_e$ . The unit vectors for the principle axes of the section (x, r, z) are  $\vec{i}, \vec{j}, \vec{k}$ ; these are for no bending, but including the elastic torsion in the pitch angle  $\theta$ . So the principle unit vectors are rotated by  $\theta$  from the HP:

$$\vec{i} = \vec{i}_e \cos \theta - \vec{k}_e \sin \theta$$

$$\vec{j} = \vec{j}_e$$

$$\vec{k} = \vec{i}_e \sin \theta + \vec{k}_e \cos \theta$$

### Description of the bending

Now the engineering beam theory assumption is introduced: iii) plane sections perpendicular to the EA remain so after the bending of the blade. Figure 3 shows the geometry of the deformed section. The deformation of the blade is described by

i) deflection of the EA:  $x_0, r_0, z_0$

ii) rotation of the section:  $\phi_1, \phi_2$

iii) twist about the EA, implicit in  $\vec{i}, \vec{k}$

The quantities  $x_0, r_0, z_0, \phi_1, \phi_2, \theta_e$  are assumed to be small.

The unit vectors of the unbent cross section are  $\vec{r}, \vec{j}, \vec{k}$ . The unit vectors in the deformed cross section are  $\vec{r}_{xs}, \vec{j}_{xs}, \vec{k}_{xs}$ :  $\vec{r}_{xs}, \vec{k}_{xs}$  are the principle axes of the section, and  $\vec{j}_{xs}$  is tangent to the deformed EA. It follows then that

$$\begin{aligned}\vec{r}_{xs} &= \vec{r} + \phi_z \vec{j} \\ \vec{j}_{xs} &= \vec{j} - \phi_z \vec{r} + \phi_x \vec{k} \\ \vec{k}_{xs} &= \vec{k} - \phi_x \vec{j}\end{aligned}$$

Now by definition  $\vec{j}_{xs} = \partial \vec{r} / \partial s$  where  $\vec{r} = x_0 \vec{r} + (r + r_0) \vec{j} + z_0 \vec{k}$  and  $s$  = arc length along the deformed EA. Hence to first order

$$\begin{aligned}\vec{j}_{xs} &= \vec{j} + (x_0 \vec{r} + z_0 \vec{k})' \\ &= \vec{j} + (x_0' + z_0 \theta') \vec{r} + (z_0' - x_0 \theta') \vec{k}\end{aligned}$$

It follows the rotation of the section is

$$\begin{aligned}-\phi_z &= x_0' + z_0 \theta' \\ \phi_x &= z_0' - x_0 \theta'\end{aligned}$$

or

$$\phi_x \vec{r} + \phi_z \vec{k} = (z_0 \vec{r} - x_0 \vec{k})'$$

The undeflected position of the blade element is

$$\vec{r} = r \vec{j} + x \vec{r} + z \vec{k}$$

and the deflected position

$$\begin{aligned}\vec{r} &= (r + r_0) \vec{j} + x_0 \vec{r} + z_0 \vec{k} + x \vec{r}_{xs} + z \vec{k}_{xs} \\ &= \underbrace{r \vec{j}}_{\text{undeflected}} + \underbrace{x_0 \vec{r} + z_0 \vec{k}}_{\text{of deflected EA}} + \underbrace{(x \phi_z - z \phi_x) \vec{j}}_{\text{section rotation}} + \underbrace{x \vec{r} + z \vec{k}}_{\text{deflected section}}\end{aligned}$$

We shall neglect  $r_0$  for now. The strain analysis is simplified then since to first order  $s = r$ ,  $r_0$  just gives uniform strain over the section, so it may be simply added back later.

### Analysis of strain

The metric of the undeformed blade -- no bending, and no torsion  
so  $\vec{\theta} = \vec{\theta}_0$  -- is

$$\begin{aligned}\vec{r} &= x\vec{i} + r\vec{j} + z\vec{k} \\ \frac{\partial \vec{r}}{\partial r} &= -x\vec{\theta}_0'\vec{i} + \vec{j} + z\vec{\theta}_0'\vec{k} \\ g_{rr} &= \frac{\partial \vec{r}}{\partial r} \cdot \frac{\partial \vec{r}}{\partial r} = 1 + \vec{\theta}_0'^2 (x^2 + z^2)\end{aligned}$$

The metric of the deformed blade, with bending and torsion, is

$$\begin{aligned}\vec{R} &= (x+x_0)\vec{i} + (r+x\phi_z - z\phi_x)\vec{j} + (z+z_0)\vec{k} \\ \frac{\partial \vec{R}}{\partial r} &= (x_0' + \theta'(z+z_0))\vec{i} + (1+x\phi_z' - z\phi_x')\vec{j} + (z_0' - \theta'(x+x_0))\vec{k} \\ G_{rr} &= \frac{\partial \vec{R}}{\partial r} \cdot \frac{\partial \vec{R}}{\partial r} = (1+x\phi_z' - z\phi_x')^2 + (x_0' + \theta'(z+z_0))^2 \\ &\quad + (z_0' - \theta'(x+x_0))^2\end{aligned}$$

Then the axial component of the strain tensor:

$$\begin{aligned}\epsilon_{rr} &= \frac{1}{2} (G_{rr} - g_{rr}) \\ &= \frac{1}{2} \left[ (1+x\phi_z' - z\phi_x')^2 - 1 + (x_0' + \theta'(z+z_0))^2 - \vec{\theta}_0'^2 z^2 \right. \\ &\quad \left. + (z_0' - \theta'(x+x_0))^2 - \vec{\theta}_0'^2 x^2 \right]\end{aligned}$$

The linear strain, for small  $x_0, z_0, \theta, \phi_x, \phi_z$ , is

$$\begin{aligned}\epsilon_{rr} \cong e_{rr} &= x\phi_z' - z\phi_x' + \theta'^2 (xx_0 + zz_0) \\ &\quad + \vec{\theta}_0' (zx_0' - xz_0' + \theta_0^2 (x^2 + z^2))\end{aligned}$$



The strain due to the blade tension,  $\epsilon_T$ , is a constant such that the tension is given by

$$T = \int_{\text{section}} E \epsilon_{rr} dA = \epsilon_T \int E dA$$

Substituting for  $\epsilon_{rr}$  and using  $\int x dA = 0$ ,  $\int x dA = x_c A$ , and

$$\int (x^2 + z^2) dA = I_p = k_p^2 A$$

where  $k_p$  is the (modulus weighted) radius of gyration about the EA, obtain for  $\epsilon_T$ :

$$\epsilon_T = \frac{T}{EA} = \phi_z'' x_c + \theta_{tw}'^2 x_0 x_c - \theta_{tw}' z_0 x_c + \theta' \theta_z' k_p^2 + r_0'$$

In this expression, the strain due to the blade extension  $r_0$  has been added. It follows the strain may be written, with  $\epsilon_T = T/EA$ :

$$\epsilon_{rr} = \epsilon_T + (x - x_c)(\phi_z'' - \theta_{tw}'^2 \phi_x) - z(\phi_x'' + \theta_{tw}' \phi_z'') + \theta_{tw}' \theta_z' (x^2 + z^2 - k_p^2)$$

#### Section moments

To find the moments on the section, the second engineering beam theory assumption is introduced: iv) all stresses except  $\sigma_{rr}$  are negligible. The axial stress is given by  $\sigma_{rr} = E \epsilon_{rr}$ . The direction of  $\sigma_{rr}$  is

$$\hat{e} = \frac{\partial \vec{R}}{\partial r} / \left| \frac{\partial \vec{R}}{\partial r} \right|$$

The moment on the deformed cross section (figure 4) is

$$\vec{M} = M_x \vec{e}_x + M_r \vec{e}_r + M_z \vec{e}_z$$

To find  $\vec{M}$ , integrate the moment about the EA due to the elemental force  $\sigma_{rr} dA$  on the cross sections:

$$\begin{aligned} d\vec{M} &= (x \vec{e}_x + z \vec{e}_z) \times \sigma_{rr} dA \\ &= [-z \vec{e}_x + x \vec{e}_z + \theta_{tw}' (x^2 + z^2) \vec{e}_r] \sigma_{rr} dA \end{aligned}$$

Integrating over the blade section, there follows the total moments due to bending and elastic torsion:

$$M_x = \int_{\text{section}} z \sigma_{rr} dA$$

$$M_z = \int_{\text{section}} x \sigma_{rr} dA$$

$$M_r = GJ \theta'_0 + \int_{\text{section}} (x^2 + z^2) \theta'_{\tau\omega} \sigma_{rr} dA$$

To  $M_r$  has been added the torsion moment  $GJ \theta'_0$ , due to shear stresses produced by elastic torsion. These moments are about the EA. For bending it is more convenient to work with moments about the tension center  $x_c$ :

$$M_x = - \int z \sigma_{rr} dA$$

$$M_z = \int (x - x_c) \sigma_{rr} dA$$

Substituting for  $\sigma_{rr}$  and integrating, the moments are:

$$M_x = E I_{zz} (\phi'_x + \theta' \phi'_z) - \theta' \theta'_0 E I_{zr}$$

$$M_z = E I_{xx} (\phi'_z - \theta' \phi'_x) + \theta' \theta'_0 E I_{xr}$$

$$M_r = (GJ + K_r^2 T + \theta'^2 E I_{rr}) \theta'_0 + \theta'_{\tau\omega} K_r^2 T + \theta' [E I_{xr} (\phi'_z - \theta' \phi'_x) - E I_{zr} (\phi'_x + \theta' \phi'_z)]$$

where

$$I_{zz} = \int z^2 dA$$

$$I_{xx} = \int (x - x_c)^2 dA$$

$$I_r = K_r^2 A = \int (x^2 + z^2) dA$$

$$I_{xr} = \int (x - x_c) (x^2 + z^2) dA$$

$$I_{zr} = \int z (x^2 + z^2) dA$$

$$I_{rr} = \int (x^2 + z^2 - K_r^2)^2 dA$$

The integrals are all over the tension carrying elements of course, i.e. modulus weighted. The tension  $T$  acts at the tension center  $x_0$ ; hence the bending moments about the EA are given from those about  $x_0$  by

$$(M_z)_{EA} = M_z + x_0 T$$

$$(M_x)_{EA} = M_x$$

The bending/torsion coupling is due to  $EI_{xp}$  and  $EI_{zp}$ ; for a symmetrical section  $EI_{zp} = 0$ .

### Vector formulation

Define the bending moment vector

$$\vec{M}_E^{(z)} = M_x \vec{r} + M_z \vec{k}$$

and the flap/lag deflection  $\vec{w} = (z_0 \vec{r} - x_0 \vec{k})$

( $\vec{M}_E^{(z)}$  is not quite the moment on the section, because  $M_x$  and  $M_z$  are really the  $\vec{r}_x$  and  $\vec{k}_x$  components of the moment). The derivatives of  $\vec{w}$  are

$$\begin{aligned} (z_0 \vec{r} - x_0 \vec{k})' &= (z_0' - x_0 \theta') \vec{r} - (x_0' + z_0 \theta') \vec{k} \\ &= \phi_x \vec{r} + \phi_z \vec{k} \end{aligned}$$

$$(z_0 \vec{r} - x_0 \vec{k})'' = (\phi_x' + \theta' \phi_z) \vec{r} + (\phi_z' - \theta' \phi_x) \vec{k}$$

Then the result for the bending and torsion moments may be written:

$$\begin{aligned} \vec{M}_E^{(z)} &= (EI_{zz} \vec{r} \vec{r} + EI_{xx} \vec{k} \vec{k}) \cdot (z_0 \vec{r} - x_0 \vec{k})'' \\ &\quad + \theta_n' \theta_c' (EI_{xp} \vec{k} - EI_{zp} \vec{r}) \end{aligned}$$

$$\begin{aligned} M_r &= [GJ + k_p^2 T + \theta_n'^2 EI_{pp}] \theta_c' + \theta_n' k_p^2 T \\ &\quad + \theta_n' (EI_{xp} \vec{k} - EI_{zp} \vec{r}) \cdot (z_0 \vec{r} - x_0 \vec{k})'' \end{aligned}$$

This is the result sought in this section.

Writing the EI dyadic as  $EI = EI_{xx} \hat{x}\hat{x} + EI_{yy} \hat{y}\hat{y}$  and the coupling as  $EI_{xy} = EI_{yx}$ , this result becomes

$$\begin{aligned} \vec{M}_E^{(2)} &= EI \vec{w}'' + \theta_{\eta}' \theta_{\epsilon}' \vec{E}I_p \\ M_T &= [GJ + k_p^2 T + \theta_{\eta}'^2 EI_{pp}] \theta_{\epsilon}' + k_p^2 T \theta_{\eta}' \\ &\quad + \theta_{\eta}' \vec{E}I_p \cdot \vec{w}'' \end{aligned}$$

This form is an obvious extension of the engineering beam theory result for uncoupled bending and torsion ( $\theta_{\eta}' = 0$  case). The vector formulation of the result is a major simplification. The vector form allows an easy transformation from one axis system to another. In fact, the vector form is independent of the axis system used (the base of the vectors), which is the source of the simplification. Working with the vector form simplifies the analysis to follow; the base of the vectors (for example, either the hub plane system,  $\hat{x}_h, \hat{y}_h$ , or the principle axis system,  $\hat{x}, \hat{y}$ ) will be considered only when come to evaluate the coefficients of the equations of motion, never in the derivation.

This is a linearized result. So the  $\hat{x}, \hat{y}$  appearing in EI and in  $\vec{w}$  are based on the trim pitch angle  $\theta = \theta^0 + \theta_{\eta}$ . The perturbation of  $\hat{x}, \hat{y}$  due to  $\theta_{\epsilon}$  gives second order moments, which have already been neglected in the derivation. The net torsion modulus is

$$GJ_{\text{net}} = GJ + k_p^2 T + \theta_{\eta}'^2 EI_{pp}$$

where  $T = \Omega^2 \int_0^l \rho r^2 dr$  = centrifugal tension in the blade. For the elastic torsion stiffness characteristic of rotor blades, the GJ term usually dominates. The  $k_p^2 T$  term is only important for very soft (torsionally) blades, near the root. The  $\theta_{\eta}'^2 EI_{pp}$  term is only important for very soft, high twist blades.

## EQUATIONS OF MOTION FOR A ROTATING BLADE

This section derives the equations of motion for a helicopter rotor blade. The blade motion considered includes coupled flap/lag bending (including the rigid modes if the blade is articulated), rigid pitch, elastic torsion, gimbal pitch and roll, and rotation speed perturbation. The analysis includes the effects of precone, droop, and sweep; feathering axis offset; and torque offset and gimbal undersling. The effects of shaft motion, and the hub forces and moments are included, so this analysis may be combined with the equations of motion for a body or support to give the complete aeroelastic model for the system. Numerous approximations are made in the course of the analysis, in order to obtain a tractable set of equations.

### Rotor Configuration

Consider an N-bladed rotor, rotating at speed  $\Omega$  (figure 5). The mth rotor blade is at

$$\psi_m = \psi + m\Delta\psi, \quad m = 1 \dots N$$

where  $\Delta\psi = 2\pi/N$  and  $\psi = \Omega\tau$  is a nondimensional time variable. The S system ( $\vec{e}_s, \vec{f}_s, \vec{k}_s$ ) is a nonrotating, hub plane coordinate system; it is an inertial frame. The B system ( $\vec{e}_b, \vec{f}_b, \vec{k}_b$ ) is a coordinate frame rotating with the mth blade. The acceleration, angular velocity, and angular acceleration of the hub, and the forces and moments exerted by the rotor on the hub are defined in the nonrotating HP frame -- the S system. The rotor blade equations of motion are derived in the rotating frame -- the B system. Figure 6(a) shows the definition of the rotor shaft motion, linear and angular displacement in an inertial frame. Figure 6(b) shows the definition of forces and moments on the hub, in the nonrotating frame.

### Blade root geometry

Figure 7 shows the blade root geometry considered (undistorted). The origin of the B system is the location of the gimbal; if there is no gimbal, this is just the point where the shaft motion and hub forces are

defined. The gimbal is at the center of the B and S frames. The hub of the rotor is  $z_{FA}$  below the gimbal (gimbal undersling). The torque offset  $x_{FA}$  is positive in the  $-z_B$  direction. The azimuth  $\psi_m$  is measured to the feathering axis line (its projection in the HP), so the FA is parallel to the  $J_B$  axis, and offset  $x_{FA}$  from the center of rotation. The precone angle  $\delta FA_1$  gives the orientation of the FA with respect to the hub plane;  $\delta FA_1$  is positive upward, and is assumed to be a small angle. The FA is offset from the center of rotation by  $r_{FA}$ ; the FA is located at  $r = r_{FA}$  along the blade. The rigid pitch rotation of the blade about the feathering axis occurs at  $r_{FA}$ . The droop angle  $\delta FA_2$  and sweep angle  $\delta FA_3$  occur at  $r_{FA}$ , just outboard of the feather bearing;  $\delta FA_2$  and  $\delta FA_3$  give the orientation of the EA of the blade with respect to the FA. Note that these angles are measured in the HP frame;  $\delta FA_2$  is positive downward, and  $\delta FA_3$  is positive aft. Both  $\delta FA_2$  and  $\delta FA_3$  are assumed to be small angles.

From the gimbal to the blade root is the hub, underslung by  $z_{FA}$  and torque offset by  $x_{FA}$ . From the root to the FA is a shank of length  $r_{FA}$ , which undistorted is a straight line at an angle  $\delta FA_1$  to the hub plane (small precone). The blade outboard of the FA at  $r_{FA}$ , undistorted, is a straight elastic axis, with small droop and sweep ( $\delta FA_2$  and  $\delta FA_3$ ) with respect to the FA direction.

From the gimbal to the root is a rigid hub. The shank (inboard of the FA at  $r_{FA}$ ) and the blade (outboard of  $r_{FA}$ ) are flexible in bending. The shank is assumed to be rigid in torsion, i.e. the effect of torsion of the hub inboard of the feathering axis is neglected. The blade outboard of the FA is flexible in torsion as well as bending. There is rigid pitch rotation of the blade about the FA, which takes place at  $r_{FA}$ , about the local direction of the FA at  $r_{FA}$ , including the bending of the shank. Inclusion of the bending flexibility of the blade inboard of the feathering axis means the the general rotor configuration is considered: the articulated rotor with the FA inboard or outboard of the hinges, or the cantilever blade with or without flexibility inboard of the FA. The special case of a rigid shank can be considered as well of course.

### Geometry of the blade

Figure 8 shows the undeformed geometry of the blade. It is assumed that i) an elastic axis exists, and the undeformed EA is a straight line; and ii) the blade has a high aspect ratio, so engineering beam theory and lifting line theory are applicable. The following notation is used:

FA		feathering axis
EA		elastic axis
CG	$x_I$	locus of section center of gravity
AC	$x_A$	locus of section aerodynamic center
TC	$x_C$	locus of section tension center

The distances  $x_I$ ,  $x_A$ , and  $x_C$  are positive aft, measured from the EA; they are in general a function of  $r$ . The corresponding  $z$  displacements are neglected, i.e. taken as zero.

The  $\tau_o, \beta_o, \bar{e}_o$  system is the EA/principle axis system of the section. The subscript "o" is for the undeformed frame, i.e. no elastic torsion in  $\Theta$ , or gimbal or rotor speed degrees of freedom. The subscript will be dropped when it is obvious what is meant. The direction of the undeformed EA is  $\beta_o$ ;  $\tau_o, \bar{e}_o$  are the directions of the local principle axes of the section, undeformed (no bending or torsion).

The span variable is  $r$ , measured from the center of rotation to the tip. This variable is dimensionless,  $r = 0$  at the root to  $r = 1$  at the tip. The section coordinates  $x$  and  $z$  are mass principle axes, with origin at the EA. It is assumed that the direction of the mass principle axes is the same as the modulus principle axes (used in engineering beam theory for the structural moments). The CG is at  $z = 0$ ,  $x = x_I$ . Usually  $x_I$  and  $x_C$  should be close. By definition then

$$\int_{\text{section}} dm = m \quad \text{section mass}$$

$$\int z dm = \int x z dm = 0$$

$$\int x dm = x_I m \quad \text{CG location}$$

and

$$\int (x^2 + z^2) dm = I_o \quad \text{section polar moment of inertia, about EA}$$

The blade pitch angle is  $\Theta$ ; here undistorted, denoted by the subscript "m". The angle  $\Theta$  is measured from the HP to the section principle axis. It is then the angle of rotation of  $\hat{z}_1 \hat{E}_0$  from the HP axes. The undeformed pitch angle is the collective plus the built in twist

$$\Theta = \Theta_m = \Theta_{coll} + \Theta_{tw}$$

where

$$\begin{aligned} \Theta_{coll} &= \text{collective pitch} \\ \Theta_{tw}(r) &= \text{twist} \end{aligned}$$

Define  $\Theta_{coll}$  as the pitch at  $r_{FA}$ , so  $\Theta_{tw}(r_{FA}^+) = 0$ . The root pitch is then  $\Theta^0 = \Theta_{coll}$ . Inboard of  $r_{FA}$ , do not have the  $\Theta_{coll}$  rotation of the blade, but there can be pitch of the local principle axes with respect to the HP, which is included in  $\Theta_{tw}$  for  $r < r_{FA}$ . Note  $\Theta_{tw}(r_{FA}^-)$  is not necessarily zero, hence there is a jump in  $\Theta$  at  $r_{FA}$  of magnitude

$$\Theta(r_{FA}^+) - \Theta(r_{FA}^-) = \Theta_{coll} - \Theta_{tw}(r_{FA}^-)$$

The trim pitch angle is then

$$\Theta = \Theta_m = \begin{cases} \Theta_{coll} + \Theta_{tw}(r) & r > r_{FA} \\ \Theta^0 = \Theta_{coll} & r = r_{FA} \\ \Theta_{tw}(r) & r < r_{FA} \end{cases}$$

It is assumed that  $\Theta_m$  is steady, constant in time, so independent of  $\Psi$ . Cyclic variations in  $\Theta$ , as may be required to trim the rotor, are included in the perturbation to the pitch angle. We shall allow the trim pitch angle to be large, hence  $\Theta_{coll}$  and  $\Theta_{tw}$  may be large angles.

The physical sweep and droop angles are defined with respect to the blade outboard of the FA, i.e. rotated by  $\Theta^0$  about the FA. Let  $\delta_{FA2}^*$  and  $\delta_{FA3}^*$  be defined with respect to the principle axes at the root (at the FA,  $r = r_{FA}$ ); these angles will be equivalent to  $\delta_{FA2}$  and  $\delta_{FA3}$  when there is zero root pitch. Hence the droop and sweep angles are



$$\begin{aligned}\delta FA_2 &= \delta FA_2^* \cos \Theta^0 + \delta FA_3^* \sin \Theta^0 \\ \delta FA_3 &= -\delta FA_2^* \sin \Theta^0 + \delta FA_3^* \cos \Theta^0\end{aligned}$$

The angles  $\delta FA_2^*$  and  $\delta FA_3^*$  are fixed geometrical constants. It follows then that  $\delta FA_2$  and  $\delta FA_3$  vary with the root pitch  $\Theta^0$ . This must be accounted for when there are perturbations to  $\Theta$  due to the rigid pitch motion of the blade. In addition, the droop and sweep only affect the blade outboard of the FA, i.e. for  $r > r_{FA}$ . This may be accounted for by including with  $\delta FA_2$  and  $\delta FA_3$  the factor  $U(r-r_{FA})$ , where

$$U(r) = \begin{cases} 1 & r > 0 \\ 0 & r < 0 \end{cases}$$

We shall follow the convention of assuming the factor  $U$  is present whenever writing  $\delta FA_2$  or  $\delta FA_3$ .

From the B system (mth blade, rotating HP axes) to the o system (undistorted EA/XS axes) there is

- 1) rotation  $\delta FA_1 - \delta FA_2$  about  $\vec{r}_B$  (small precone and droop)
- 2) rotation  $\delta FA_3$  about  $\vec{k}_B$  (small sweep)
- 3) then rotation  $\Theta_m$  about  $\vec{j}_{EA}$  (large pitch angle)

Hence

$$\begin{aligned}\vec{r}_o &= \cos \Theta_m \vec{r}_B - \sin \Theta_m \vec{k}_B + j_B [(\delta FA_1 - \delta FA_2) \sin \Theta_m - \delta FA_3 \cos \Theta_m] \\ \vec{k}_o &= \sin \Theta_m \vec{r}_B + \cos \Theta_m \vec{k}_B + j_B [-(\delta FA_1 - \delta FA_2) \cos \Theta_m - \delta FA_3 \sin \Theta_m] \\ \vec{j}_o &= \vec{j}_{EA} = j_B + \delta FA_3 \vec{r}_B + (\delta FA_1 - \delta FA_2) \vec{k}_B\end{aligned}$$

where  $\delta FA_2$  and  $\delta FA_3$  are based on  $\Theta_m^0 = \Theta_{L0}$ , and are absent for  $r < r_{FA}$ . We shall drop the subscripts "o" and "m", denoting the trim and undistorted geometry, when it is obvious what is meant.

### Motion

The rotor blade motion (degrees of freedom of the rotor) is described by:

- 1) gimbal motion (optional); pitch and roll of the rotor disk.
- 2) rotor speed perturbation.
- 3) Then elastic torsion about the EA, and rigid pitch about the FA.
- 4) Followed by bending deflection of the FA, including rigid flap and lag motion if the blade is articulated.

#### Gimbal motion/rotor speed perturbation

Figure 9(a) shows the gimbal motion and rotor speed perturbation in the nonrotating frame. The gimbal degrees of freedom are  $\beta_{0c}$  and  $\beta_{0s}$ : rotation of the rotor disk, in the nonrotating frame (S system), with the same convention as  $\beta_{1c}$  and  $\beta_{1s}$  tip path plane tilt. The rotor rotational speed perturbation is  $\dot{W}_s$ . The degree of freedom  $W_s$  is a rotation about the shaft axis  $\vec{k}_s$ ; so the azimuth angle of the  $m$ th blade is really  $\psi_m + W_s$ .

Figure 9(b) shows the gimbal motion in the rotating frame. The degrees of freedom are  $\beta_0$  and  $\Theta_0$ , given by

$$\begin{aligned}\beta_0 &= \beta_{0c} \cos \psi_m + \beta_{0s} \sin \psi_m \\ \Theta_0 &= -\beta_{0c} \sin \psi_m + \beta_{0s} \cos \psi_m\end{aligned}$$

The main effects are due to  $\beta_0$ , the flapwise rotation about the  $\vec{r}_g$  axis;  $\Theta_0$ , the rotation about  $\vec{j}_g$ , only introduces a translation of the hub due to  $x_{FA}$  and  $y_{FA}$ . The blade pitch  $\Theta$  is defined with respect to the hub plane, so only the blade inboard of the FA sees the pitch rotation due to  $\Theta_0$ , and that effect will be neglected.

#### Blade motion

Figure 3 shows the geometry of the deformed blade. The blade deformation is described by:

- 1) twist about the EA:  $\Theta$
- 2) deflection of the EA:  $x_0, z_0$
- 3) rotation of the section:  $\phi_x, \phi_z$

The pitch angle  $\Theta$ , including perturbations, is implicit in the  $\vec{r}, \vec{j}, \vec{k}$  system;  $\vec{r}, \vec{k}$  are the principle axes of the blade with no bending, but now with the blade elastic torsion and rigid pitch motion in  $\Theta$ . The XS system ( $\vec{r}_{xs}, \vec{j}_{xs}, \vec{k}_{xs}$ ) are the principle axes and EA of the deformed blade, including torsion and bending. The tangent to the deformed EA is  $\vec{j}_{xs}$ ; the rotation of the cross section from  $\vec{r}, \vec{k}$  is given by  $\phi_x$  and  $\phi_z$ :

$$\begin{aligned}\phi_x \vec{r} + \phi_z \vec{k} &= (\vec{r}_0' - x_0 \Theta') \vec{r} - (x_0' + z_0 \Theta') \vec{k} \\ &= (\vec{r}_0' - x_0 \vec{k}')'\end{aligned}$$

The blade position, relative the root, is then:

$$\begin{aligned}\vec{r} &= (r + r_0) \vec{j} + x_0 \vec{r} + z_0 \vec{k} + x \vec{r}_{xs} + z \vec{k}_{xs} \\ &= \underbrace{(r + r_0 + x \phi_z - z \phi_x)}_{\text{radial station}} \vec{j} + \underbrace{(x_0 \vec{r} + z_0 \vec{k})}_{\text{axial extension}} + \underbrace{x \vec{r} + z \vec{k}}_{\text{rotation of section}} + \underbrace{x \vec{r}_{xs} + z \vec{k}_{xs}}_{\text{elastic bending}} + \underbrace{x \vec{r}_{xs} + z \vec{k}_{xs}}_{\text{pitch twist}}\end{aligned}$$

We will neglect the perturbation of the radial position,  $r_0 + x \phi_z - z \phi_x \ll r$ .

### Blade pitch

The angle  $\Theta$  is the angle of the major principle axis of the section (the x axis, chordwise), measured from the hub plane. The blade pitch is composed of:

- 1)  $\Theta^0(\psi)$  = root pitch, the pitch of the blade at the FA at  $r = r_{FA}^+$ ; due to commanded collective and control, control system flexibility, and mechanical feedback.
- 2)  $\Theta_{in}(r) = \text{built-in twist; } \Theta_{in}(r_{FA}^+) = 0.$
- 3)  $\Theta_e(r, \psi) = \text{elastic torsion about the EA; zero at the FA, } \Theta_e(r_{FA}, \psi) = 0; \text{ only } \Theta_e \text{ produces torsion shear stress in the blade.}$

For the shank,  $r < r_{FA}$ , elastic torsion is neglected, and it does not see the root pitch  $\theta^o$ . Then  $\theta_{tw}(r)$  is used for the pitch of the principle axes with respect to the hub plane in the shank. There is no perturbation to  $\theta$  inboard of the FA, the pitch and torsion degrees of freedom are only for outboard of the FA. Since probably  $\theta_{tw}(r_{FA})$  is not zero, there is a jump in  $\theta$  at the FA. So the blade pitch is

$$\theta = \begin{cases} \theta^o + \theta_{tw} + \theta_c & r > r_{FA} \\ \theta^o & r = r_{FA}^+ \\ \theta_{tw} & r < r_{FA} \end{cases}$$

The commanded root pitch angle is

$$\theta^c = \theta_{coll} + \theta_{con}$$

where

$\theta_{coll}$  = collective pitch angle; the trim value, which may be large but is assumed to be steady in time.

$\theta_{con}$  = control input; time dependent, but assumed to be a small angle; includes cyclic to trim the rotor; and for dynamics analyses this is the control variable.

The blade root pitch commanded by the control system is  $\theta^c$ ;  $\theta^o$  is the actual blade root pitch. The difference  $(\theta^o - \theta^c)$  is the rigid pitch motion due to control system flexibility or mechanical coupling in the control system (i.e.  $\delta_j$  effects). Hence we may write the blade pitch as:

$$\theta = \begin{cases} (\theta_{coll} + \theta_{tw}) + (\theta^o - \theta^c) + \theta_{con} + \theta_c & r > r_{FA} \\ \theta^o = \theta_{coll} + (\theta^o - \theta^c) + \theta_{con} & r = r_{FA}^+ \\ \theta_{tw} & r < r_{FA} \end{cases}$$

Now the pitch  $\theta$  may be separated into trim and perturbation contributions:

$$\theta = \begin{cases} \theta_m + \tilde{\theta} & r > r_{FA} \\ \theta_m^o + \tilde{\theta}^o & r = r_{FA}^+ \\ \theta_m & r < r_{FA} \end{cases}$$

where the trim terms are (as above)

$$\vartheta_m = \begin{cases} \vartheta_{coll} + \vartheta_{tw} \\ \vartheta_{coll} \\ \vartheta_{tw} \end{cases}$$

and the perturbations

$$\tilde{\vartheta} = \begin{cases} (\vartheta^o - \vartheta^c) + \vartheta_{coll} + \vartheta_e \\ \tilde{\vartheta}^o = (\vartheta^o - \vartheta^c) + \vartheta_{coll} \\ 0 \end{cases}$$

The trim value of the pitch is  $\vartheta_m$ , composed of  $\vartheta_{coll}$  and  $\vartheta_{tw}$ ; it is a large, steady angle. The perturbation of the pitch angle is  $\tilde{\vartheta}$ , composed of the blade motion  $(\vartheta^o - \vartheta^c)$ ,  $\vartheta_{coll}$  and  $\vartheta_e$ ; all are small angles, so  $\tilde{\vartheta}$  is small. For the rigid pitch degree of freedom we shall use  $p_o$ , defined as

$$p_o = \tilde{\vartheta}^o = (\vartheta^o - \vartheta^c) + \vartheta_{coll}$$

and for the elastic pitch  $\vartheta_e$  an expansion as a series in the normal modes (described in the sections to follow). Note that  $p_o$  is the total rigid pitch perturbation, including the control  $\vartheta_{coll}$ .

## Coordinate Frames and Axes

S System: nonrotating, hub plane frame

rotation  $\psi_m - 90$  about  $\vec{k}_s$

B system: rotating, mth blade, hub plane

$\beta_0$  about  $\vec{e}_B$

$\theta_0$  about  $\vec{j}_B$

$\psi_s$  about  $\vec{k}_B$

H system: hub frame

$\delta FA_1$  about  $\vec{e}_H$

FA system: blade FA (EA for  $r < r_{FA}$ )

$-\delta FA_2$  about  $\vec{e}_{FA}$

$-\delta FA_3$  about  $\vec{k}_{FA}$

EA system: EA outboard of FA

$\Theta$  about  $\vec{j}_{EA}$

$-\theta_0$  about  $\vec{j}_{EA}$

blade system: principle axes, including torsion

$\phi_x$  about  $\vec{e}$

$\phi_z$  about  $\vec{k}$

XS system: principle axes, torsion and bending

### B system

$$\vec{e}_B = \sin \psi_m \vec{e}_s - \cos \psi_m \vec{j}_s$$

$$\vec{j}_B = \cos \psi_m \vec{e}_s + \sin \psi_m \vec{j}_s$$

$$\vec{k}_B = \vec{k}_s$$

### Blade system

From the B system to the blade system, there is first rotation  $\beta_0 + \delta FA_1, -\delta FA_2$  about  $\vec{e}_B$  and rotation  $\psi_s - \delta FA_3$  about  $\vec{k}_B$ ; then rotation  $\Theta$  about  $\vec{j}_{EA}$ . Hence:

$$\begin{aligned}
\vec{r} &= \cos \Theta \vec{r}_B - \sin \Theta \vec{k}_B \\
&\quad + \vec{j}_B [(\beta_G + \delta FA_1 - \delta FA_2) \sin \Theta + (\psi_s - \delta FA_3) \cos \Theta] \\
\vec{k} &= \sin \Theta \vec{r}_B + \cos \Theta \vec{k}_B \\
&\quad + \vec{j}_B [-(\beta_G + \delta FA_1 - \delta FA_2) \cos \Theta + (\psi_s - \delta FA_3) \sin \Theta] \\
\vec{j} &= \vec{j}_{EA} = \vec{j}_B - (\psi_s - \delta FA_3) \vec{r}_B + (\beta_G + \delta FA_1 - \delta FA_2) \vec{k}_B
\end{aligned}$$

For  $r < r_{FA}$ , the  $\delta FA_2$  and  $\delta FA_3$  terms drop; in particular:

$$\vec{j} = \vec{j}_{FA} = \vec{j}_B - \psi_s \vec{r}_B + (\beta_G + \delta FA_1) \vec{k}_B$$

#### XS system

$$\begin{aligned}
\vec{r}_{XS} &= \vec{r} + \phi_x \vec{j} \\
\vec{j}_{XS} &= \vec{j} - \phi_x \vec{r} + \phi_x \vec{k} = \vec{j} + (\kappa_0 \vec{r} + \tau_0 \vec{k})' \\
\vec{k}_{XS} &= \vec{k} - \phi_x \vec{j}
\end{aligned}$$

#### Undisturbed blade system

The undisturbed blade system is  $\vec{r}, \vec{j}, \vec{k}$  without  $(\beta_G, \psi_s)$ , or the pitch perturbations in  $\Theta$  (and  $\delta FA_2$  &  $\delta FA_3$  based on  $\Theta_m = \Theta_{un}$ ); hence:

$$\begin{aligned}
\vec{r}_0 &= \cos \Theta_m \vec{r}_B - \sin \Theta_m \vec{k}_B + \vec{j}_B [(\delta FA_1 - \delta FA_2) \sin \Theta_m - \delta FA_3 \cos \Theta_m] \\
\vec{k}_0 &= \sin \Theta_m \vec{r}_B + \cos \Theta_m \vec{k}_B + \vec{j}_B [(\delta FA_1 - \delta FA_2) \cos \Theta_m - \delta FA_3 \sin \Theta_m] \\
\vec{j}_0 &= \vec{j}_B + \delta FA_3 \vec{r}_B + (\delta FA_1 - \delta FA_2) \vec{k}_B
\end{aligned}$$

Now since the blade motion  $\tilde{\Theta}$ ,  $\beta_G$ , and  $\psi_s$  is small, it is possible to expand the blade system in terms of the undisturbed frame:

$$\begin{aligned}
\vec{r} &\approx \vec{r}_0 - \tilde{\Theta} \vec{k}_0 + \vec{j}_0 [(\beta_G - \tilde{\Theta}' \delta FA_2) \sin \Theta + (\psi_s + \tilde{\Theta}' \delta FA_2) \cos \Theta] \\
\vec{k} &= \vec{k}_0 + \tilde{\Theta} \vec{r}_0 + \vec{j}_0 [-(\beta_G - \tilde{\Theta}' \delta FA_2) \cos \Theta + (\psi_s + \tilde{\Theta}' \delta FA_2) \sin \Theta]
\end{aligned}$$

There follows then:

$$\begin{aligned}
 (x_0 \vec{e} + z_0 \vec{k}) &= (x_0 \vec{e}_0 + z_0 \vec{k}_0) + \tilde{\Theta} (z_0 \vec{e}_0 - x_0 \vec{k}_0) \\
 &+ J_B [(\psi_s + \tilde{\Theta} \delta FA_2) \vec{e}_B - (\beta_G - \tilde{\Theta} \delta FA_1) \vec{k}_B] \cdot (x_0 \vec{e}_0 + z_0 \vec{k}_0)
 \end{aligned}$$

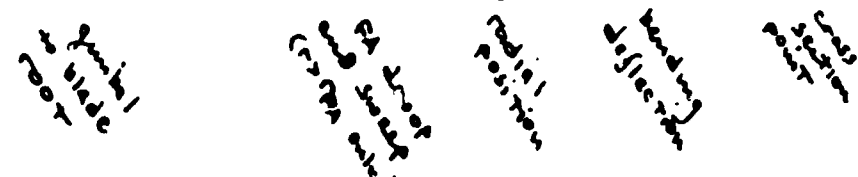
which is an expansion of the bending/torsion deflection of the blade in terms of the undisturbed axis system.

### Blade position, velocity, and acceleration

#### Position

The distance from the gimbal to a point on the blade section is

$$\vec{r} = -z_{FA} \vec{e}_H - x_{FA} \vec{e}_H + r_{FA} (J_{FA} - J_{BA}) + r \vec{J} + x_0 \vec{e} + z_0 \vec{k} + x \vec{e}_{x5} + z \vec{k}_{x5}$$



which may be written

$$\begin{aligned}
 \vec{r} &= \vec{e}_B (-x_{FA} - z_{FA} \Theta_G - r_{FA} \delta FA_3) \\
 &+ J_B (z_{FA} \beta_G - x_{FA} \psi_s) \\
 &+ \vec{k}_B (-z_{FA} + x_{FA} \Theta_G + r_{FA} \delta FA_2) \\
 &+ r \vec{J} + (x_0 \vec{e} + z_0 \vec{k}) + (x \vec{e} + z \vec{k}) \\
 \\ 
 &= \vec{e}_B (-x_{FA} - z_{FA} \Theta_G - r_{FA} \psi_s + (r - r_{FA}) \delta FA_3) \\
 &+ J_B (r + z_{FA} \beta_G - x_{FA} \psi_s) \\
 &+ \vec{k}_B (-z_{FA} + x_{FA} \Theta_G + r(\beta_G + \delta FA_1) - (r - r_{FA}) \delta FA_2) \\
 &+ (x_0 \vec{e} + z_0 \vec{k}) + (x \vec{e} + z \vec{k})
 \end{aligned}$$



### Velocity

The velocity of a point on the blade, relative to the rotating frame (the B system) is:

$$\begin{aligned}\vec{v}_r = \left( \frac{\partial}{\partial t} \vec{r} \right)_B &= \tau_B (-z_{FA} \dot{\theta}_G - r \dot{\psi}_S) \\ &+ j_B (z_{FA} \dot{\beta}_G - x_{FA} \dot{\psi}_S) \\ &+ \vec{k}_B (x_{FA} \dot{\theta}_G + r \dot{\beta}_G) \\ &+ (r - r_{FA}) \dot{\theta}^0 (-\tau_B \delta FA_2 - \vec{k}_B \delta FA_3) \\ &+ ((x_0 + x) \vec{i} + (z_0 + z) \vec{k})^{\cdot}\end{aligned}$$

where

$$\begin{aligned}((x_0 + x) \vec{i} + (z_0 + z) \vec{k})^{\cdot} &\cong (x_0 \vec{i}_0 + z_0 \vec{k}_0)^{\cdot} \\ &+ \dot{\theta} ((z_0 + z) \vec{i}_0 - (x_0 + x) \vec{k}_0)\end{aligned}$$

### Acceleration

The acceleration of a point on the blade, relative to the rotating frame, and neglecting the squares of velocities, is:

$$\begin{aligned}\vec{a}_r = \left( \frac{\partial}{\partial t} \vec{v}_r \right)_B &= \tau_B (-z_{FA} \ddot{\theta}_G - r \ddot{\psi}_S) \\ &+ j_B (z_{FA} \ddot{\beta}_G - x_{FA} \ddot{\psi}_S) \\ &+ \vec{k}_B (x_{FA} \ddot{\theta}_G + r \ddot{\beta}_G) \\ &+ (r - r_{FA}) \ddot{\theta}^0 (-\tau_B \delta FA_2 - \vec{k}_B \delta FA_3) \\ &+ ((x_0 + x) \vec{i} + (z_0 + z) \vec{k})^{\cdot\cdot}\end{aligned}$$

where

$$\begin{aligned}((x_0 + x) \vec{i} + (z_0 + z) \vec{k})^{\cdot\cdot} &\cong (x_0 \vec{i}_0 + z_0 \vec{k}_0)^{\cdot\cdot} \\ &+ \ddot{\theta} ((z_0 + z) \vec{i}_0 - (x_0 + x) \vec{k}_0)\end{aligned}$$

### Acceleration of the blade

The acceleration of the blade is required with respect to an inertia frame, i.e. in the S system. The B system rotates at a constant angular velocity  $\vec{\Omega} = \Omega \vec{k}_B$  with respect to the S frame. The shaft motion is composed of linear and angular velocity and acceleration of

the origin of the S frame (the gimbal point at the hub center of rotation). The acceleration, angular velocity, and angular acceleration of the S system, with respect to the nonrotating, inertial frame, are:

$$\begin{aligned}\vec{a}_0 &= \ddot{x}_0 \vec{e}_s + \ddot{y}_0 \vec{e}_s + \ddot{z}_0 \vec{k}_s \\ \vec{\omega}_0 &= \dot{\omega}_x \vec{e}_s + \dot{\omega}_y \vec{e}_s + \dot{\omega}_z \vec{k}_s \\ \dot{\vec{\omega}}_0 &= \ddot{\omega}_x \vec{e}_s + \ddot{\omega}_y \vec{e}_s + \ddot{\omega}_z \vec{k}_s\end{aligned}$$

It is assumed that  $\vec{a}_0$ ,  $\vec{\omega}_0$ , and  $\dot{\vec{\omega}}_0$  are all small quantities.

Given above is the motion of the blade in the B frame, the acceleration and velocity of the blade  $\vec{a}_r$  and  $\vec{v}_r$ . Now we shall derive the acceleration of a blade point in inertial space ( $\vec{a}$ ), in terms of the motion of the shaft, the rotation of the rotor, and the blade motion in the B frame. From the result for the acceleration in a rotating coordinate frame, there follows:

$$\vec{a} = \vec{a}_0 + \vec{a}_{r,s} + 2\vec{\omega}_0 \times \vec{v}_{r,s} + \vec{\omega}_0 \times (\vec{\omega}_0 \times \vec{r}) + \dot{\vec{\omega}}_0 \times \vec{r}$$

where  $\vec{a}_{r,s}$  and  $\vec{v}_{r,s}$  are the acceleration and velocity of a point in the S frame. The B system rotates at angular velocity  $\vec{\Omega} = \Omega \vec{k}_s$  with respect to the S frame. Hence with  $\Omega$  constant and no angular acceleration or acceleration of B with respect to S, there follows:

$$\begin{aligned}\vec{a}_{r,s} &= \vec{a}_r + 2\vec{\Omega} \times \vec{v}_r + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \\ \vec{v}_{r,s} &= \vec{v}_r + \vec{\Omega} \times \vec{r}\end{aligned}$$

where  $\vec{a}_r$  and  $\vec{v}_r$  are the acceleration and velocity in the B frame. Thus:

$$\begin{aligned}\vec{a} &= \vec{a}_0 + \vec{a}_r + 2\vec{\Omega} \times \vec{v}_r + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \\ &\quad + 2\vec{\omega}_0 \times \vec{v}_r + 2\vec{\omega}_0 \times (\vec{\Omega} \times \vec{r}) + \vec{\omega}_0 \times (\vec{\omega}_0 \times \vec{r}) + \dot{\vec{\omega}}_0 \times \vec{r}\end{aligned}$$

To first order in the velocity and angular velocity, this becomes:

$$\begin{aligned}\vec{a} &\approx \vec{a}_0 + \vec{a}_r + 2\vec{\Omega} \times \vec{v}_r + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \\ &\quad + 2\vec{\omega}_0 \times (\vec{\Omega} \times \vec{r}) + \dot{\vec{\omega}}_0 \times \vec{r}\end{aligned}$$

The six terms are respectively: the acceleration of the origin; the relative acceleration in the rotating frame; the relative coriolis acceleration; the centrifugal acceleration; the coriolis acceleration due to the angular velocity of the origin; and the angular acceleration of the origin. In dyadic operator form, and with  $\vec{\omega} = \omega \vec{k}_0$ , this result is

$$\vec{a} = \vec{a}_0 + \vec{a}_r + 2\omega (\vec{E}_0 \times) \vec{v}_r - \omega^2 (\vec{r}_0 \vec{r}_0 + \vec{j}_0 \vec{j}_0) \vec{r} + 2\omega (\vec{E}_0 \vec{r} - \vec{r} \vec{E}_0) \vec{\omega}_0 - (\vec{r} \times) \dot{\vec{\omega}}_0$$

To obtain the forces and moments and equations of motion, the acceleration is multiplied by the density of the blade point (  $\rho dr$  ) and integrated over the volume of the blade, to produce the total acceleration of the blade.

### Equations of Motion and Forces

The equations of motion for elastic bending, torsion, and rigid pitch of the blade are obtained from equilibrium of inertial, aerodynamic, and elastic moments on the portion of the blade outboard of  $r$ :

$$-\vec{M}_E + \vec{M}_A = \vec{M}_I$$

where

$M_E$  = structural moment on deformed cross section, on the inboard face; so  $-M_E$  is the external force on the outboard face.

$M_A$  = total aerodynamic force on blade surface outboard of  $r$ .

$M_I$  = total acceleration of the blade outboard of  $r$ .

$M_E$  is a general elastic constraint: from engineering beam theory for bending and torsion; from control system flexibility for rigid pitch; hub spring for global motion; or it is the force or moment on the hub due to the rotor (so  $-M_E$  is the force on the rotor).  $M_I$  is the angular acceleration of the blade outboard of  $r$ , about the point  $\vec{r}_0(r)$ :

$$\vec{M}_I = \int_r^l \int_{\text{section}} (\vec{r}(s) - \vec{r}_0(r)) \times \vec{a} \, dm \, ds$$

For bending, engineering beam theory gives

$$\vec{M}_E^{(2)} = M_x \vec{e}_x + M_z \vec{e}_z = (\vec{e}_x \vec{e}_{xs} + \vec{e}_z \vec{e}_{zs}) \vec{M}_E$$

So this operator is applied to  $\vec{M}_I$  and  $\vec{M}_A$  also. For bending the moments about the tension center  $x_C$  are required. Then the desired PDE for bending is obtained from  $\frac{\partial^2}{\partial r^2} \vec{M}^{(2)}$ .

For elastic torsion, engineering beam theory gives  $M_{rE} = \vec{J}_{xs} \cdot \vec{M}_E$ . So this same operator is applied to  $\vec{M}_I$  and  $\vec{M}_A$ . For torsion require moments about the section EA ( $x = 0$ ) at  $r$ ; also, elastic torsion involves only  $r > r_{FA}$ . The desired PDE for torsion is then obtained from  $\frac{\partial}{\partial r} M_r$ .

The equation of motion for rigid pitch degree of freedom  $p_0 = \tilde{\theta}$  is obtained from equilibrium of moments about the FA:

$$M_{FA} = \vec{J}_{xs}(\vec{r}_{FA}) \cdot \vec{M}(\vec{r}_{FA})$$

where  $\vec{M}$  is the moment about the FA ( $x = 0$ ) at  $r = r_{FA}$ . The elastic restraint from the control system flexibility gives the restoring moment about the FA, completing the desired equation of motion.

The equations of motion for the gimbal degrees of freedom ( $\beta_{gc}$  and  $\beta_{gs}$ ) are obtained from equilibrium of moments about the gimbal:

$$M_x = \vec{r}_s \cdot \vec{M}$$

$$M_y = \vec{r}_s \cdot \vec{M}$$

where  $\vec{M}$  is the total moments (from all  $N$  blades) about the gimbal point, in the nonrotating frame.

The equation of motion for the speed perturbation degree of freedom  $\psi_s$  is obtained from equilibrium of torque moments  $\mathcal{Q} = -M_s = \vec{r}_s \cdot \vec{M}$  where again  $\vec{M}$  is the total moment about the gimbal point.

The total rotor force and moment on the hub (at the gimbal point) are obtained from a sum over the  $N$  blades of  $\vec{F}^{(m)}$  and  $\vec{M}^{(m)}$ , the force and moment due to the  $m$ th blade:

$$\vec{F} = \sum_{m=1}^N \vec{F}^{(m)}$$

$$\vec{M} = \sum_{m=1}^N \vec{M}^{(m)}$$

Since  $-\vec{F}^{(m)}$  and  $-\vec{M}^{(m)}$  are the forces on the blade, there follows from force and moment equilibrium of the entire blade:

$$-\vec{F}^{(m)} + \vec{F}_A = \vec{F}_x$$

$$-\vec{M}^{(m)} + \vec{M}_A = \vec{M}_x$$

The hub force and moment are required in the nonrotating hub plane frame (the S system); the components are defined as:

$$\vec{F} = H\vec{e}_s + Y\vec{e}_s + T\vec{k}_s$$

$$\vec{M} = M_x\vec{e}_s + M_y\vec{e}_s - Q\vec{k}_s$$

Note  $\vec{M}$  produces the gimbal and rotor speed perturbation motion, if those degrees of freedom are used, but it is also transmitted through the gimbal to the helicopter body or support.

### Aerodynamics

The aerodynamic forces and moments on the blade are obtained from the integral over the span of the aerodynamic forces and pitch moments on the blade section. The aerodynamic forces and moment on the section are:

$F_x$	in hub plane, positive in drag direction, ( $\vec{e}_s$ direction), at the EA
$F_z$	normal to the hub plane, positive up ( $\vec{k}_s$ direction), at the EA
$F_r$	radial, positive outward ( $\vec{e}_s$ direction), at the EA
$M_a$	moment about the EA, positive nose up

The forces on the section are  $F_x$ ,  $F_z$ , and  $F_r$ ; these are the component of the aerodynamic lift and drag forces in the hub plane axis system (the B frame).  $F_r$  is here just the radial drag force; the radial components

of  $F_x$  and  $F_z$  due to tilt of the blade when it is bent are included explicitly in the results below.

The aerodynamic force on the section, at the deformed EA, including the effect of the rotation of the section due to bending, is thus:

$$\vec{F}_{aero} = F_x \vec{e}_B + F_z \vec{e}_B - \vec{J}_B \vec{J}_{xs} \cdot (F_x \vec{e}_B + F_z \vec{e}_B) + F_r \vec{J}_{xs}$$

$$\cong F_x \vec{e}_B + F_z \vec{e}_B + \vec{J}_B (F_r - F_z (\beta_0 + \delta PA_1 - \delta PA_2 + \vec{k}_B \cdot (\vec{x}_0 \vec{e}_t + \vec{z}_0 \vec{e}_k)^v))$$

and the aerodynamic moment:

$$\vec{M}_{aero} = M_a \vec{J}_{xs}$$

# Equations of Motion and Hub Forces/Moments

## Bending

The equation of motion comes from

$$\frac{\partial^2}{\partial r^2} \vec{M}_E^{(z)} + \frac{\partial^2}{\partial r^2} \vec{M}_I^{(z)} = \frac{\partial^2}{\partial r^2} \vec{M}_A^{(z)}$$

where  $\vec{M}$  is the moment about the tension center ( $x = x_0$ ) at  $r$ , and

$$\vec{M}^{(z)} = (\vec{r} \times \vec{r}_c + \vec{r}_c \times \vec{r}_c) \vec{M} = (\vec{r} \times \vec{r} + \vec{r}_c \times \vec{r}_c - (x_0 \vec{r} + z_0 \vec{r}_c) \cdot \vec{r}) \vec{M}$$

Inertia: Considering first  $r > r_{FA}$ , the moment is

$$\begin{aligned} \vec{M}_I &= \int_r^1 \int_{\text{section}} (\vec{r} \times \vec{r}_c - \vec{r} \times \vec{r}_{c0}) \times \vec{a} \, d\mu \, d\varphi \\ &= \int_r^1 \int \left[ (y-r) \vec{j} + (x_0+x) \vec{i} + (z_0+z) \vec{k} \right. \\ &\quad \left. - ((x_0+x_c) \vec{i} + z_0 \vec{k}) \right] \times \vec{a} \, d\mu \, d\varphi \end{aligned}$$

So

$$\begin{aligned} \frac{\partial \vec{M}}{\partial r} &= -(\vec{j} + (x_0 \vec{i} + z_0 \vec{k} + x_c \vec{i}) \cdot \vec{r}) \times \int_r^1 \int \vec{a} \, d\mu \, d\varphi \\ &\quad - \int (x \vec{i} + z \vec{k} - x_c \vec{i}) \times \vec{a} \, d\mu \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \vec{M}}{\partial r^2} &= \vec{r} \times \int \vec{a} \, d\mu \\ &\quad - [(x_0 \vec{i} + z_0 \vec{k} + x_c \vec{i}) \cdot \vec{r} \times \int_r^1 \int \vec{a} \, d\mu \, d\varphi] \cdot \vec{r} \\ &\quad - [\int (x \vec{i} + z \vec{k} - x_c \vec{i}) \times \vec{a} \, d\mu] \cdot \vec{r} \end{aligned}$$

$$\vec{j} \cdot \vec{M} = \int_r^1 \int \left[ (z_0+z) \vec{i} - (x_0+x) \vec{k} \right. \\ \left. - (z_0 \vec{i} - x_0 \vec{k} - x_c \vec{k}) \right] \cdot \vec{a} \, d\mu \, d\varphi$$

So

$$\begin{aligned}\frac{\partial^2}{\partial r^2} \vec{M}_I^{(2)} &= (\vec{r}\vec{r} + \vec{k}\vec{k}) \frac{\partial^2 \vec{M}_I}{\partial r^2} - [(\vec{x}_0\vec{r} + \vec{z}_0\vec{k})^\vee \vec{J} \cdot \vec{M}_I]^\vee \\ &= \vec{J} \times \int \vec{\alpha} d\Omega \\ &\quad + \left[ \int (\vec{z}\vec{r} - \vec{x}\vec{k} + \vec{x}_c\vec{k}) \vec{J} \cdot \vec{\alpha} d\Omega \right]^\vee \\ &\quad + \left[ (\vec{z}_0\vec{r} - \vec{x}_0\vec{k} - \vec{x}_c\vec{k})^\vee \int_r^1 \int \vec{J} \cdot \vec{\alpha} d\Omega d\varphi \right]^\vee \\ &\quad - \left[ (\vec{x}_0\vec{r} + \vec{z}_0\vec{k})^\vee \int_r^1 \left[ (\vec{z}_0 + \vec{r})\vec{r} - (\vec{x}_0 + \vec{x})\vec{k} \right. \right. \\ &\quad \left. \left. - (\vec{z}_0\vec{r} - \vec{x}_0\vec{k} - \vec{x}_c\vec{k}) \right]_r \cdot \vec{\alpha} d\Omega d\varphi \right]^\vee\end{aligned}$$

We shall neglect the last term in this result,  $[(\vec{x}_0\vec{r} + \vec{z}_0\vec{k})^\vee \vec{J} \cdot \vec{M}_I]^\vee$ , as order  $(c/R)^2$  smaller than the first term. Including the case  $r < r_{FA}$ , which only introduces an effect of droop and sweep, the result is:

$$\begin{aligned}\frac{\partial^2}{\partial r^2} \vec{M}_I^{(2)} &= \vec{J} \times \int \vec{\alpha} d\Omega \\ &\quad + \left[ \int (\vec{z}\vec{r} - \vec{x}\vec{k} + \vec{x}_c\vec{k}) \vec{J} \cdot \vec{\alpha} d\Omega \right]^\vee \\ &\quad + \left[ (\vec{z}_0\vec{r} - \vec{x}_0\vec{k} - \vec{x}_c\vec{k})^\vee \int_r^1 \int \vec{J} \cdot \vec{\alpha} d\Omega d\varphi \right]^\vee \\ &\quad - \delta(r - r_{FA}) (\delta FA_2 \vec{r}_0 + \delta FA_3 \vec{k}_0) \int_{r_{FA}}^1 \int \vec{J} \cdot \vec{\alpha} d\Omega d\varphi\end{aligned}$$

where  $\delta(r)$  is the delta function, i.e. an impulse at  $r=0$ .

a) shaft motion: with  $\vec{r} \approx r \vec{j}_0$  have

$$\begin{aligned}\vec{\alpha} &= \vec{\alpha}_0 + 2\Omega (\vec{k}_0 \vec{r} - \vec{r} \vec{k}_0) \vec{\omega}_0 - \vec{r} \times \vec{\omega}_0 \\ &\approx \vec{\alpha}_0 + 2\Omega r (\vec{k}_0 \vec{j}_0 - \vec{j}_0 \vec{k}_0) \vec{\omega}_0 - r \vec{j}_0 \times \vec{\omega}_0\end{aligned}$$

So

$$\begin{aligned}\frac{\partial^2 \vec{M}_I^{(2)}}{\partial r^2} &\approx m \vec{j}_0 \times \vec{\alpha} = m (\vec{r}_0 \vec{k}_0 - \vec{k}_0 \vec{r}_0) \vec{\omega}_0 \\ &\quad + 2\Omega r m (\vec{r}_0 \vec{j}_0) \vec{\omega}_0 \\ &\quad + m r (\vec{r}_0 \vec{r}_0 + \vec{k}_0 \vec{k}_0) \vec{\omega}_0\end{aligned}$$



b) relative acceleration:

$$\frac{\partial^2 \vec{M}^{(2)}}{\partial r^2} \approx \int \times \int \vec{a}_r \, d\mu$$

$$= m \left[ \begin{array}{l} -\vec{E}_B (-z_{FA} \ddot{\Theta}_G - r \ddot{\Psi}_3) \\ + \vec{E}_B (x_{FA} \ddot{\Theta}_G + r \ddot{\beta}_G) \\ + (z_0 \vec{r} - x_0 \vec{k})'' \\ - \ddot{\Theta} (x_0 + x_z) \vec{r} + z_0 \vec{k} \\ - \ddot{\Theta} (r - r_{FA}) (\delta_3 \vec{E}_B - \delta_2 \vec{E}_B) \end{array} \right]$$

c) centrifugal acceleration:

$$\vec{a} = -\Omega^2 (\vec{r}_B \vec{r}_B + \int_B \int_B) \vec{r}$$

$$\text{so } \int \times \vec{a} = \Omega^2 \vec{E}_B [-x_{FA} - z_{FA} \Theta_G - r_{FA} \delta_3 + \vec{r}_B \cdot ((x_0 + x) \vec{r} + (z_0 + z) \vec{k})] \\ + \Omega^2 \vec{r}_B r (\beta_G + \delta_1 - \delta_2)$$

$$\int \cdot \vec{a} \approx -\Omega^2 r$$

so

$$\frac{\partial^2 \vec{M}^{(2)}}{\partial r^2} = -\Omega^2 \left[ \begin{array}{l} [(z_0 \vec{r} - x_0 \vec{k})' \int_r' \int^m \Omega_g] + m \vec{E}_B \vec{E}_B \cdot (z_0 \vec{r} - x_0 \vec{k}) \\ - [(\ddot{\Theta} (x_0 \vec{r} + z_0 \vec{k} + x_z \vec{r}))' \int_r' \int^m \Omega_g] \\ + [(x_c - x_z) \ddot{\Theta} \vec{r} m]' - m \vec{E}_B \ddot{\Theta} \vec{E}_B \cdot (x_0 \vec{r} + z_0 \vec{k} + x_z \vec{r}) \\ - \delta(r - r_{FA}) \ddot{\Theta} (\delta_3 \vec{r}_B - \delta_2 \vec{E}_B) \int_{r_{FA}}' \int^m \Omega_g \\ + \ddot{\Theta} m [r \delta_3 \vec{r}_B - r_{FA} \delta_2 \vec{E}_B] \\ + \vec{E}_B m z_{FA} \Theta_G - \vec{r}_B m r \beta_G \end{array} \right] \\ - \Omega^2 \left[ \begin{array}{l} [(x_c - x_z) \vec{k} r m]' - [(x_c \vec{k})' \int_r' \int^m \Omega_g] \\ - \delta(r - r_{FA}) (\delta_2 \vec{r}_B + \delta_3 \vec{E}_B) \int_{r_{FA}}' \int^m \Omega_g \\ + \vec{E}_B m [x_{FA} + r_{FA} \delta_3 - x_z \cos \Theta] \\ - \vec{r}_B m r (\delta_1 - \delta_2) \end{array} \right]$$

d) coriolis accelerations:

$$\vec{a} = 2\Omega \vec{k}_B \times \vec{v}_r$$

$$\text{So } \dot{\vec{r}} \cdot \vec{a} = 2\Omega \left\{ -r\dot{\psi}_s - \vec{k}_B \cdot (\dot{z}_0\vec{e} - x_0\dot{\vec{e}}) \right\}$$

$$\dot{\vec{r}} \times \vec{a} = 2\Omega \left\{ \vec{k}_B \dot{\vec{r}} + [-(\delta_1 - \delta_2)\vec{e}_B + \delta_3\vec{k}_B] [-r\dot{\psi}_s - \vec{k}_B \cdot (\dot{z}_0\vec{e} - x_0\dot{\vec{e}})] \right\}$$

For  $\dot{\vec{r}} \cdot \vec{v}_r$ , it is here necessary to include the effect of the change in the radial position of the blade due to bending:

$$\Delta \vec{r} = -\dot{\vec{r}}_B \frac{1}{2} \int_0^r \left[ (x_0\vec{e} + z_0\vec{k} + x_{\pm}\vec{e})' - (\psi_s - \delta_3)\vec{e}_B + (\beta_G + \delta_1 - \delta_2)\vec{k}_B \right]^2 dy$$

$$\begin{aligned} \text{So } \dot{\vec{r}} \cdot \vec{v}_r &= - \int_0^r (\dot{z}_0\vec{e} - x_0\dot{\vec{e}})' \cdot (\dot{z}_0\vec{e} - x_0\dot{\vec{e}} - x_{\pm}\dot{\vec{e}})' dy \\ &\quad - (\dot{z}_0\vec{e} - x_0\dot{\vec{e}})' \cdot ((\delta_1 - \delta_2)\vec{e}_B - \delta_3\vec{k}_B) \\ &\quad - \beta_G [-z_{FA} + \vec{e}_B \cdot (\dot{z}_0\vec{e} - x_0\dot{\vec{e}} - x_{\pm}\dot{\vec{e}}) + r\delta_1 - (r-r_{FA})\delta_2] \\ &\quad - \dot{\psi}_s [x_{FA} + \vec{k}_B \cdot (\dot{z}_0\vec{e} - x_0\dot{\vec{e}} - x_{\pm}\dot{\vec{e}}) - (r-r_{FA})\delta_3] \end{aligned}$$

then

$$\frac{\partial^2 \vec{M}^{(2)}}{\partial r^2} = 2\Omega \left[ \begin{aligned} &-\vec{k}_B m \int_0^r (\dot{z}_0\vec{e} - x_0\dot{\vec{e}})' \cdot (\dot{z}_0\vec{e} - x_0\dot{\vec{e}} - x_{\pm}\dot{\vec{e}})' dy \\ &-\vec{k}_B m \beta_G [-z_{FA} + \vec{e}_B \cdot (\dot{z}_0\vec{e} - x_0\dot{\vec{e}} - x_{\pm}\dot{\vec{e}}) + r\delta_1 - (r-r_{FA})\delta_2] \\ &-\vec{k}_B m \dot{\psi}_s [x_{FA} + \vec{k}_B \cdot (\dot{z}_0\vec{e} - x_0\dot{\vec{e}} - x_{\pm}\dot{\vec{e}}) + r_{FA}\delta_3] \\ &- [(x_c - x_{\pm})\vec{k} m (r\dot{\psi}_s + \vec{k}_B \cdot (\dot{z}_0\vec{e} - x_0\dot{\vec{e}})')] \cdot \\ &- [(\dot{z}_0\vec{e} - x_0\dot{\vec{e}} - x_c\dot{\vec{e}})' \int_r^1 (\dot{\psi}_s + \vec{k}_B \cdot (\dot{z}_0\vec{e} - x_0\dot{\vec{e}})') m dy]' \\ &+ \delta(r-r_{FA})(\delta_2\vec{e}_B + \delta_3\vec{k}_B) \int_{r_{FA}}^1 (\dot{\psi}_s + \vec{k}_B \cdot (\dot{z}_0\vec{e} - x_0\dot{\vec{e}})') m dy \\ &+ m(\delta_1 - \delta_2) \dot{\vec{r}} \times [(\dot{z}_0\vec{e} - x_0\dot{\vec{e}})' + r\dot{\psi}_s \vec{k}_B] \end{aligned} \right]$$

elastic:

$$\frac{\partial^2 \vec{M}_E^{(2)}}{\partial r^2} = \left[ (EI_{zz} \tau \tau + EI_{xx} \xi \xi) (\tau_0 \tau - x_0 \xi)'''' \right]'' + \left[ (EI_{xr} \xi - EI_{zr} \tau) \theta_m' \theta_e' \right]''$$

aerodynamic: The moment about the tension center ( $x = x_0$ ) at  $r$  due to the blade loading at the EA at  $g$ :

$$\vec{M}_A = \int_r^1 (\vec{r}|_{g00} - \vec{r}|_{rx0}) \times \vec{F}_{aero} dg$$

$$\cong \int_r^1 (g-r) (F_z \tau_0 - F_x \xi_0) dg$$

So

$$\frac{\partial^2 \vec{M}_A^{(2)}}{\partial r^2} = \int_x \vec{F}_{aero} = F_z \tau_0 - F_x \xi_0$$

### Elastic torsion

The equation of motion is obtained from

$$-\frac{\partial}{\partial r} M_{rE} - \frac{\partial}{\partial r} M_{rT} = -\frac{\partial}{\partial r} M_{rA}$$

where  $\vec{M}$  is the moment about the EA at  $r$ , and

$$\frac{\partial}{\partial r} M_r = \frac{\partial}{\partial r} \int_{xs} \vec{M} = \int \cdot \frac{\partial \vec{M}}{\partial r} + [(\tau_0 \tau + \xi_0 \xi)' \cdot \vec{M}]'$$

inertia:

$$\vec{M}_I = \int_r^1 \int_{section} (\vec{r}|_{g00} - \vec{r}|_{r00}) \times \vec{a} dm dg$$

$$= \int_r^1 \int \left[ (g-r) \vec{J} + (\tau_0 + \tau) \tau + (\xi_0 + \xi) \xi - (\tau_0 \tau + \xi_0 \xi) \tau \right] \times \vec{a} dm dg$$

So

$$\frac{\partial \vec{M}}{\partial r} = -(\vec{J} + (\tau_0 \tau + \xi_0 \xi)') \times \int_r^1 \int \vec{a} dm dg - \int (\tau \tau + \xi \xi) \times \vec{a} dm$$

so

$$\begin{aligned} \frac{\partial M_{rz}}{\partial r} = & \int (x\vec{E} - z\vec{t}) \cdot \vec{a} \, d\omega - (z_0\vec{t} - x_0\vec{E})^{\vee\vee} \cdot \int_r^1 (r-\rho) \int \vec{a} \, d\omega \, d\rho \\ & - (x_0\vec{t} + z_0\vec{E})^{\vee} \cdot \int (x\vec{E} - z\vec{t}) \cdot \vec{a} \, d\omega \\ & - (x_0\vec{t} + z_0\vec{E})^{\vee\vee} \cdot \int_r^1 \int \left[ \begin{array}{c} (z_0+z)\vec{t} - (x_0+x)\vec{E} \\ -(z_0\vec{t} - x_0\vec{E})/r \end{array} \right] \cdot \vec{a} \, d\omega \, d\rho \end{aligned}$$

The ODE for the  $k$ th torsion node of the  $m$ th blade is obtained by operating with  $\int_{r_{FA}}^1 \xi_k(\dots) dr$ ; where  $\xi_k$  is the elastic torsion mode shape. It is most convenient to apply this operator at this point:

$$\begin{aligned} \int_{r_{FA}}^1 \xi_k \frac{\partial M_{rz}}{\partial r} dr = & \int_{r_{FA}}^1 \int \left\{ \xi_k (x\vec{E} - z\vec{t}) \right. \\ & \left. - \int_{r_{FA}}^r \xi_k (z_0\vec{t} - x_0\vec{E})^{\vee\vee} (r-\rho) d\rho \right\} \cdot \vec{a} \, d\omega \, dr \\ & - \int_{r_{FA}}^1 \xi_k \left\{ \begin{array}{l} (x_0\vec{t} + z_0\vec{E})^{\vee} \cdot \int (x\vec{E} - z\vec{t}) \cdot \vec{a} \, d\omega \\ + (x_0\vec{t} + z_0\vec{E})^{\vee\vee} \cdot \int_r^1 \left[ \begin{array}{c} (z_0+z)\vec{t} - (x_0+x)\vec{E} \\ -(z_0\vec{t} - x_0\vec{E})/r \end{array} \right] \cdot \vec{a} \, d\omega \, d\rho \end{array} \right\} dr \end{aligned}$$

and we shall use the notation:

$$\vec{X}_k = \xi_k x_{\perp} \vec{E} - \int_{r_{FA}}^r \xi_k (z_0\vec{t} - x_0\vec{E})^{\vee\vee} (r-\rho) d\rho$$

a) shaft motion:

$$\begin{aligned} \int_{r_{FA}}^1 \xi_k \frac{\partial M_{rz}}{\partial r} dr = & \left( \int_{r_{FA}}^1 \vec{X}_k \, m \, dr \right) (\vec{t}_B \vec{t}_B + \vec{E}_B \vec{E}_B) \vec{\omega} \\ & + 2S_L \left( \int_{r_{FA}}^1 \vec{X}_k \, r \, m \, dr \right) \cdot \vec{E}_B \cdot \vec{J}_B \cdot \vec{\omega} \\ & + \left( \int_{r_{FA}}^1 \vec{X}_k \, r \, m \, dr \right) (\vec{E}_B \vec{t}_B - \vec{t}_B \vec{E}_B) \vec{\omega}_0 \end{aligned}$$

b) relative acceleration:

$$\begin{aligned}
 \int_{r_{FA}}^1 \xi_k \frac{\partial M_r}{\partial r} dr = & \left( \int_{r_{FA}}^1 \bar{X}_k m dr \right) \cdot (-\ddot{z}_{FA} \ddot{\theta}_G \tau_B + x_{FA} \ddot{\theta}_G \ddot{E}_B) \\
 & + \left( \int_{r_{FA}}^1 \bar{X}_k r m dr \right) \cdot (-\ddot{\psi}_s \tau_B + \beta_G \ddot{E}_B) \\
 & + \int_{r_{FA}}^1 \bar{X}_k \cdot (x_0 \ddot{\tau} + z_0 \ddot{E})'' m dr \\
 & + \int_{r_{FA}}^1 \left[ \bar{X}_k \cdot (z_0 \ddot{\tau} - x_0 \ddot{E} - x_z \ddot{E}) + \xi_k x_z^2 \right] \ddot{\theta} m dr \\
 & - \int_{r_{FA}}^1 \bar{X}_k \cdot (\delta_2 \ddot{\tau}_B + \delta_3 \ddot{E}_B) (r - r_{FA}) \ddot{\theta}^0 m dr \\
 & - \int_{r_{FA}}^1 \xi_k \ddot{\theta} \mathcal{I}_\theta dr
 \end{aligned}$$

where  $\mathcal{I}_\theta = \int x^2 + z^2 dm =$  section pitch moment of inertia, about EA.

c) centrifugal acceleration: neglecting a number of terms due to blade torsion and pitch (of the same order as the propeller moment), compared to the structural stiffening; there follows:

$$\int_{r_{FA}}^1 \xi_k \frac{\partial M_r}{\partial r} dr = -\Omega^2 \left[ \begin{aligned}
 & - \left( \int_{r_{FA}}^1 \bar{X}_k m dr \right) \cdot \tau_B \tau_{FA} \theta_G \\
 & - \left( \int_{r_{FA}}^1 \bar{X}_k r m dr \right) \cdot \tau_B \beta_G \\
 & + \int_{r_{FA}}^1 \xi_k \ddot{\theta} \mathcal{I}_\theta (\cos^2 \theta - \sin^2 \theta) dr \\
 & - \int_{r_{FA}}^1 \bar{X}_k \cdot \ddot{E}_B \ddot{E}_B \cdot (x_0 \ddot{\tau} + z_0 \ddot{E} + x_z \ddot{\tau}) m dr \\
 & - \int_{r_{FA}}^1 \xi_k \left\{ (x_0 \ddot{\tau} + z_0 \ddot{E})'' \cdot \ddot{E} x_z m \right. \\
 & \quad \left. - (x_0 \ddot{\tau} + z_0 \ddot{E})'' \cdot (z_0 \ddot{\tau} - x_0 \ddot{E}) \int_r^1 \xi_m dm \right\} dr \\
 & + \int_{r_{FA}}^1 \xi_k \left\{ (x_0 \ddot{\tau} + z_0 \ddot{E}) \cdot \ddot{E} x_z m \right. \\
 & \quad \left. - r (x_0 \ddot{\tau} + z_0 \ddot{E})'' \cdot \int_r^1 (z_0 \ddot{\tau} - x_0 \ddot{E} - x_z \ddot{E}) m dm \right\} dr \\
 & + \int_{r_{FA}}^1 \bar{X}_k \cdot [\tau_B (-x_{FA} - r_m \delta_3) - \ddot{E}_B r (\delta_1 - \delta_2)] m dr
 \end{aligned} \right]$$

elastic:

using

$$T = \Omega^2 \int_r^1 g^m dq$$

$$\begin{aligned} \frac{\partial M_{rE}}{\partial r} = & \left[ (GJ + k_r^2 \Omega^2 \int_r^1 g^m dq + \Theta_{rw}^2 EI_{rp}) \Theta_e'' \right]' \\ & + (\Theta_{rw}^2 k_r^2 \Omega^2 \int_r^1 g^m dq)' \\ & + \left[ \Theta_{rw}^2 (EI_{rp} \bar{K} - EI_{zp} \bar{L}) \cdot (z_0 \bar{L} - x_0 \bar{K})'' \right]' \end{aligned}$$

aerodynamic: the moment about the EA at r is

$$\vec{M}_A = \int_r^1 M_a \vec{j}_{xs} dq + \int_r^1 \begin{bmatrix} (g-r) \vec{j} + (x_0 \bar{L} + z_0 \bar{K}) \\ -(x_0 \bar{L} + z_0 \bar{K}) \bar{i}_r \end{bmatrix} \times \vec{F}_{aero} dq$$

$$\text{so } \frac{\partial \vec{M}_A}{\partial r} = -M_a \vec{j}_{xs} - \vec{j}_{xs} \times \int_r^1 \vec{F}_{aero} dq$$

$$\begin{aligned} \frac{\partial M_{rA}}{\partial r} &= \frac{\partial}{\partial r} \vec{j}_{xs} \cdot \vec{M}_A = \vec{j}_{xs} \cdot \frac{\partial \vec{M}_A}{\partial r} + (x_0 \bar{L} + z_0 \bar{K})'' \cdot \vec{M}_A \\ &= -M_a - (z_0 \bar{L} - x_0 \bar{K})'' \cdot \int_r^1 (g-r) (F_x \bar{L}_0 + F_z \bar{K}_0) dq \end{aligned}$$

thus

$$\int_{r_{FA}}^1 \vec{j}_x \frac{\partial M_{rA}}{\partial r} dr = - \int_{r_{FA}}^1 \vec{j}_k M_a dr + \int_{r_{FA}}^1 (F_x \bar{L}_0 + F_z \bar{K}_0) \cdot \vec{X}_{Ak} dr$$

where

$$\vec{X}_{Ak} = \vec{X}_k - \vec{j}_k \times \bar{r}_k$$

### Rigid pitch

The equation of motion comes from  $M_{FAE} + M_{FAI} = M_{FAA}$ , where

$$M_{FA} = \int_{V_A} (r_{FA}) \cdot \vec{M} = (\int_{FA} + (x_0 \vec{e}_1 + z_0 \vec{e}_3) |_{r_{FA}}) \cdot \vec{M}$$

and  $\vec{M}$  is the moment about the FA, at  $r = r_{FA}$ .

inertia:

$$\begin{aligned} \vec{M}_I &= \int_{r_{FA}}^1 \left( \vec{r} |_{r_{FA}} - \vec{r} |_{r_{FA} \infty} \right) \times \vec{\omega} \, d\mu \, d\mu \\ &= \int_{r_{FA}}^1 \left[ \begin{aligned} (r - r_{FA}) \vec{e}_1 + (x_0 + x) \vec{e}_2 + (z_0 + z) \vec{e}_3 \\ - (x_0 \vec{e}_1 + z_0 \vec{e}_3) |_{r_{FA}} \end{aligned} \right] \times \vec{\omega} \, d\mu \, d\mu \end{aligned}$$

So

$$\begin{aligned} M_{FAI} &= \int_{r_{FA}}^1 \left[ \begin{aligned} (z_0 + z) \vec{e}_1 - (x_0 + x) \vec{e}_2 - (\delta_2 \vec{e}_2 + \delta_3 \vec{e}_3) (r - r_{FA}) \\ - (z_0 \vec{e}_1 - x_0 \vec{e}_2) |_{r_{FA}} - (z_0 \vec{e}_1 - x_0 \vec{e}_2) |_{r_{FA}} (r - r_{FA}) \end{aligned} \right] \cdot \vec{\omega} \, d\mu \, d\mu \\ &+ \int_{r_{FA}}^1 \left[ \begin{aligned} - (x_0 \vec{e}_1 + z_0 \vec{e}_3) |_{r_{FA}} \cdot \left\{ \begin{aligned} (z_0 + z) \vec{e}_1 - (x_0 + x) \vec{e}_2 \\ - (z_0 \vec{e}_1 - x_0 \vec{e}_2) |_{r_{FA}} \end{aligned} \right\} \\ + (\delta_2 \vec{e}_2 - \delta_3 \vec{e}_3) \cdot \left\{ \begin{aligned} (z_0 + z) \vec{e}_1 - (x_0 + x) \vec{e}_2 \\ + (z_0 \vec{e}_1 - x_0 \vec{e}_2) |_{r_{FA}} (r - r_{FA}) \end{aligned} \right\} \end{aligned} \right] \cdot \vec{\omega} \, d\mu \, d\mu \end{aligned}$$

and we shall use the notation:

$$\begin{aligned} \vec{\chi}_0 &= - (z_0 \vec{e}_1 - x_0 \vec{e}_2 - x_2 \vec{e}_2) + (\delta_2 \vec{e}_2 + \delta_3 \vec{e}_3) (r - r_{FA}) \\ &+ (z_0 \vec{e}_1 - x_0 \vec{e}_2) |_{r_{FA}} + (z_0 \vec{e}_1 - x_0 \vec{e}_2) |_{r_{FA}} (r - r_{FA}) \end{aligned}$$

a) shaft motion:

$$\begin{aligned} M_{FA} &= - \left( \int_{r_{FA}}^1 \vec{\chi}_0 \, d\mu \, d\mu \right) (\vec{e}_2 \vec{e}_2 + \vec{e}_3 \vec{e}_3) \vec{\omega}_0 \\ &- 2 \Omega \left( \int_{r_{FA}}^1 \vec{\chi}_0 \, d\mu \, d\mu \right) \cdot \vec{e}_3 \, \vec{e}_3 \, \vec{\omega}_0 \\ &- \left( \int_{r_{FA}}^1 \vec{\chi}_0 \, d\mu \, d\mu \right) (\vec{e}_3 \vec{e}_2 - \vec{e}_2 \vec{e}_3) \vec{\omega}_0 \end{aligned}$$

b) relative acceleration:

$$\begin{aligned}
 M_{FA} = & - \left( \int_{r_{FA}}^1 \vec{X}_0 m dr \right) \cdot (-z_{FA} \ddot{\theta}_G \vec{e}_\theta + x_{FA} \ddot{\theta}_G \vec{e}_\theta) \\
 & - \left( \int_{r_{FA}}^1 \vec{X}_0 r m dr \right) \cdot (-\ddot{\psi}_G \vec{e}_\theta + \beta_G \vec{e}_\theta) \\
 & - \int_{r_{FA}}^1 \vec{X}_0 \cdot (x_0 \vec{e} + z_0 \vec{e})'' m dr \\
 & - \int_{r_{FA}}^1 \left[ \vec{X}_0 \cdot (z_0 \vec{e} - x_0 \vec{e} - x_z \vec{e}) + x_z^2 \right] \ddot{\theta} m dr \\
 & + \int_{r_{FA}}^1 \vec{X}_0 \cdot (\delta_2 \vec{e}_\theta + \delta_3 \vec{e}_\theta) (r - r_{FA}) \ddot{\theta} m dr \\
 & + \int_{r_{FA}}^1 \ddot{\theta} I_\theta dr
 \end{aligned}$$

c) centrifugal acceleration:

$$M_{FA} = -\omega^2 \left[ \begin{aligned}
 & \left( \int_{r_{FA}}^1 \vec{X}_0 m dr \right) \cdot \vec{e}_\theta z_{FA} \theta_G \\
 & + \left( \int_{r_{FA}}^1 \vec{X}_0 r m dr \right) \cdot \vec{e}_\theta \beta_G \\
 & - \int_{r_{FA}}^1 \ddot{\theta} I_\theta (\cos^2 \theta - \sin^2 \theta) dr \\
 & - \int_{r_{FA}}^1 \vec{X}_0 \cdot [\vec{e}_\theta (-x_{FA} - r_{FA} \delta_3) - \vec{e}_\theta r (\delta_1 - \delta_2)] m dr \\
 & + \int_{r_{FA}}^1 \vec{X}_0 \cdot \vec{e}_\theta \vec{e}_\theta \cdot (x_0 \vec{e} + z_0 \vec{e} + x_z \vec{e}) m dr \\
 & + (x_0 \vec{e} + z_0 \vec{e})' |_{r_{FA}} \cdot (z_0 \vec{e} - x_0 \vec{e}) |_{r_{FA}} \int_{r_{FA}}^1 r m dr \\
 & + (x_0 \vec{e} + z_0 \vec{e})' |_{r_{FA}} \cdot (\delta_2 \vec{e}_\theta + \delta_3 \vec{e}_\theta) \int_{r_{FA}}^1 r (r - r_{FA}) m dr \\
 & - \int_{r_{FA}}^1 (x_0 \vec{e} + z_0 \vec{e} + x_z \vec{e}) \cdot \begin{bmatrix} (z_0 \vec{e} - x_0 \vec{e}) |_{r_{FA}} \\ -r_{FA} (z_0 \vec{e} - x_0 \vec{e})' |_{r_{FA}} \\ -r_{FA} (\delta_2 \vec{e}_\theta + \delta_3 \vec{e}_\theta) \end{bmatrix} m dr
 \end{aligned} \right]$$



aerodynamics: moment about the FA at  $r_{FA}$  is

$$\vec{M}_A = \int_{r_{FA}}^1 M_a \vec{j}_{xs} dr + \int_{r_{FA}}^1 \begin{bmatrix} (r-r_{FA})\vec{j} + (x_0\vec{i} + z_0\vec{k}) \\ -(x_0\vec{i} + z_0\vec{k})/r_{FA} \end{bmatrix} \times \vec{r}_{aero} dr$$

So

$$M_{FAA} = \int_{r_{FA}}^1 M_a dr - \int_{r_{FA}}^1 (F_x \vec{e}_B + F_z \vec{k}_B) \cdot \vec{X}_{A_0} dr$$

where

$$\vec{X}_{A_0} = \vec{X}_0 - x_{\pm} \vec{k}$$

elastic:

The aerodynamic and inertial moments about the FA are reacted by moments due to the deformation of the control system, due to commanded pitch angle, and due to feedback (mechanical or kinematic) from the blade bending or gimbal motion. The restoring moment about the feathering axis on the blade is  $-M_{con}$ ; it is given by the control system flexibility, i.e. the elastic deformation in the control system  $\Theta_{ec}$  times the control system stiffness  $K_{con}$ . Hence:

$$M_{con} = K_{con} \Theta_{ec} = K_{con} (\delta^0 - \Theta_{con} + \sum_i k_{p_i} q_i + k_{p_G} \beta_G)$$

The  $q_i$  are the bending degrees of freedom, so  $K_p$  are the pitch/flap and pitch/lag coupling, mechanical or kinematic feedback due to the control system and blade root geometry. Similarly,  $K_{p_G}$  is the pitch/flap coupling for the gimbal motion. For the rigid flap motion of the blade, this coupling is given by the  $\delta_j$  angle, such that  $K_p = \tan \delta_j$ . For a rigid control system,  $K_{con} \rightarrow \infty$ , the rigid pitch equation of motion reduces to

$$p_0 = \delta^0 \rightarrow \Theta_{con} = \sum_i k_{p_i} q_i + k_{p_G} \beta_G$$

So  $p_0$  becomes just the control input, and pitch/bending coupling.

Now we write the control system stiffness  $K_{con}$  in terms of the nonrotating natural frequency of the rigid pitch motion of the blade,  $\omega_o$  :

$$K_{con} = \left( \int_{r_{FA}}^1 I_{\theta} dr \right) \omega_o^2$$

Then:

$$M_{FAE} = M_{con} = \left( \int_{r_{FA}}^1 I_{\theta} dr \right) \omega_o^2 \left[ p_c - \theta_{con} + \sum K_r q_i + K_{\beta} \beta_G \right]$$

### Force

The net force of the mth blade on the hub is

$$\vec{F}^{(m)} = \vec{F}_A - \vec{F}_Z$$

where  $\vec{F}$  is the force due to the blade, at the hub.

inertia: 
$$\vec{F}_Z = \int_0^1 \int_{section} \vec{a} dm dr$$

a) shaft motion:

$$\vec{F} = \left( \int_0^1 m dr \right) \ddot{\alpha}_o + 2\Omega \left( \int_0^1 r m dr \right) (\vec{e}_B \dot{J}_B - J_B \dot{\vec{e}}_B) \dot{\omega}_o + \left( \int_0^1 r m dr \right) (\vec{e}_B \dot{\tau}_B - \dot{\tau}_B \vec{e}_B) \dot{\omega}_o$$

b) relative acceleration:

$$\vec{F} = \left( \int_0^1 r m dr \right) (-\dot{\tau}_B \ddot{\psi}_s + \vec{e}_B \ddot{\beta}_G) + \int_0^1 (x_o \dot{\tau} + \tau_o \dot{\vec{e}})'' m dr$$

c) coriois acceleration:

$$\begin{aligned} \vec{F} &= 2\Omega \int_0^1 \int \vec{e}_B \times \vec{v}_r dm dr \\ &= 2\Omega J_B \left[ - \left( \int_0^1 r m dr \right) \dot{\psi}_s + \int_0^1 \dot{\tau}_B \cdot (x_o \dot{\tau} + \tau_o \dot{\vec{e}})' m dr \right] \end{aligned}$$

d) centrifugal acceleration:

$$\begin{aligned}\vec{F} &= -\Omega^2 \int_0^1 \int (\tau_B \tau_B + j_B j_B) \vec{r} \, dm \, dr \\ &= -\Omega^2 \left[ \tau_B \left[ \int_0^1 (-x_{FA} + (r - r_{FA}) \delta_3) \, m \, dr \right] \right. \\ &\quad \left. + j_B \left( \int_0^1 r \, m \, dr \right) - \tau_B \left( \int_0^1 r \, m \, dr \right) \psi_3 \right. \\ &\quad \left. + \tau_B \int_0^1 \tau_B \cdot (x_0 \tau + z_0 \tau + x_z \tau) \, m \, dr \right]\end{aligned}$$

aerodynamics:

$$\begin{aligned}\vec{F}_A &= \int_0^1 \vec{F}_{aero} \, dr \\ &= \int_0^1 [F_x \tau_B + F_z \tau_B \\ &\quad + j_B (F_r - F_z (\beta_0 + \delta_1 - \delta_2 + \vec{k}_B \cdot (x_0 \tau + z_0 \tau)^\circ))] \, dr\end{aligned}$$

### Moment

The net moment of the mth blade on the hub, about the gimbal point,

is:

$$\vec{M}^{(m)} = \vec{M}_A - \vec{M}_H$$

inertia:

$$\vec{M}_H = \int_0^1 \int \vec{r} \times \vec{a} \, dm \, dr$$

a) shaft motion:

$$\begin{aligned}\vec{M} &= \left( \int_0^1 r \, m \, dr \right) (\tau_B \vec{k}_B - \vec{k}_B \tau_B) \vec{\omega}_0 \\ &\quad + 2\Omega \left( \int_0^1 r^2 \, m \, dr \right) \tau_B j_B \cdot \vec{\omega}_0 \\ &\quad + \left( \int_0^1 r^2 \, m \, dr \right) (\tau_B \tau_B + \vec{k}_B \vec{k}_B) \vec{\omega}_0\end{aligned}$$

b) relative acceleration:

$$\vec{M} = \left[ \begin{aligned} & (\int_0^1 r m dr) (\ddot{z}_{FA} \vec{k}_B + \gamma_{FA} \ddot{\theta}_G \vec{z}_B) \\ & + (\int_0^1 r^2 m dr) (\ddot{k}_B \vec{\psi}_s + \ddot{z}_B \vec{\beta}_G) \\ & + \int_0^1 (\ddot{z}_0 \vec{z} - \ddot{x}_0 \vec{k}) r m dr \\ & - \int_{r_{FA}}^1 \ddot{\theta} ((x_0 + x_z) \vec{z} + z_0 \vec{k}) r m dr \\ & - \ddot{\theta}^0 (\delta_3 \vec{z}_B - \delta_2 \vec{k}_B) \int_{r_{FA}}^1 (r - r_{FA}) r m dr \end{aligned} \right]$$

c) centrifugal acceleration:

$$\begin{aligned} \vec{r} \times (\ddot{\vec{z}} \times (\vec{\omega} \times \vec{r})) &= \vec{r} \times \ddot{\vec{z}} \vec{\omega} \cdot \vec{r} = -\omega^2 \vec{k}_B \times \vec{r} \vec{k}_B \cdot \vec{r} \\ &\cong -\omega^2 r (-\vec{z}_B \vec{k}_B \cdot \vec{r}) \end{aligned}$$

So

$$\vec{M} = \omega^2 \vec{z}_B \left[ \begin{aligned} & \int_0^1 (-z_{FA} + r \delta_1 - (r - r_{FA}) \delta_2) r m dr \\ & + (\int_0^1 r^2 m dr) \beta_G \\ & + \int_0^1 \vec{k}_B \cdot (x_0 \vec{z} + z_0 \vec{k} + x_z \vec{z}) r m dr \\ & + \int_{r_{FA}}^1 \ddot{\theta} \vec{k}_B \cdot (z_0 \vec{z} - x_0 \vec{k} - x_z \vec{k}) r m dr \\ & - \ddot{\theta}^0 \delta_{FA3} \left( \int_{r_{FA}}^1 (r - r_{FA}) r m dr \right) \end{aligned} \right]$$

d) coriolis acceleration:

$$\vec{r} \times \vec{\alpha} = 2\Omega \left[ \vec{j}_B \times \vec{r} (r\dot{\psi}_s + \vec{k}_B \cdot (z_0\vec{e} - x_0\vec{k}))' \right. \\ \left. + r\vec{k}_B \vec{j}_B \cdot \vec{v}_r \right]$$

and

$$\vec{j}_B \times \vec{r} = -\vec{k}_B (-x_{FA} + (r-r_{FA})\delta_3) + \vec{e}_B (-z_{FA} + r\delta_1 - (r-r_{FA})\delta_2) \\ + (z_0\vec{e} - x_0\vec{k} - x_z\vec{k})$$

So

$$\vec{M} = 2\Omega \left[ \vec{e}_B \int_0^1 (r\dot{\psi}_s + \vec{k}_B \cdot (x_0\vec{e} + z_0\vec{k}))' \left[ \begin{array}{l} -z_{FA} + r\delta_1 - (r-r_{FA})\delta_2 \\ + \vec{e}_B \cdot (z_0\vec{e} - x_0\vec{k} - x_z\vec{k}) \end{array} \right] \omega dr \right. \\ \left. + \vec{k}_B \left[ \begin{array}{l} - \int_0^1 \left( \int_0^r (z_0\vec{e} - x_0\vec{k})' \cdot (z_0\vec{e} - x_0\vec{k} - x_z\vec{k})' d\eta \right) r \omega dr \\ + \int_0^1 \vec{e}_B \cdot (z_0\vec{e} - x_0\vec{k})' (-r\delta_1 + (r-r_{FA})\delta_2) \omega dr \\ + \int_0^1 \vec{k}_B \cdot (z_0\vec{e} - x_0\vec{k})' (x_{FA} + r_{FA}\delta_3) \omega dr \\ + \int_0^1 \vec{k}_B \cdot (z_0\vec{e} - x_0\vec{k})' \vec{k}_B \cdot (z_0\vec{e} - x_0\vec{k} - x_z\vec{k}) \omega dr \\ - \dot{\beta}_B \int_0^1 \left[ -z_{FA} + \vec{e}_B \cdot (z_0\vec{e} - x_0\vec{k} - x_z\vec{k}) \right. \\ \left. + r\delta_1 - (r-r_{FA})\delta_2 \right] r \omega dr \end{array} \right]$$

aerodynamics:

$$\vec{M}_A = \int_0^1 \vec{r} \times \vec{F}_{aero} dr \\ \cong \int_0^1 (F_z \vec{e}_B - F_x \vec{k}_B) dr$$

### Gimbal

The equations of motion for the gimbal degrees of freedom are obtained from the  $\tau_s$  &  $j_s$  components of  $\vec{M} = \sum \vec{M}^{(n)}$ ; thus:

$$\vec{M}_{HS} + \vec{M}_I = \vec{M}_A$$

where  $\vec{M}_{HS}$  is the spring and damper moment at the gimbal, reacting the rotor applied moment. The gimbal spring and damper are assumed to be in the nonrotating frame. Hence:

$$\begin{aligned} \vec{M}_{HS} = & \tau_s (k_G \beta_{Gs} + c_G \dot{\beta}_{Gs}) \\ & - j_s (k_G \beta_{Gc} + c_G \dot{\beta}_{Gc}) \end{aligned}$$

Taking the  $\tau_s$  and  $j_s$  components of  $\vec{M}$ , the gimbal equations of motion are:

$$\begin{aligned} M_y + c_G \dot{\beta}_{Gc} + k_G \beta_{Gc} &= 0 \\ -M_x + c_G \dot{\beta}_{Gs} + k_G \beta_{Gs} &= 0 \end{aligned}$$

We shall write the gimbal hub spring and damper as:

$$k_G = \frac{N}{2} I_0 \Omega^2 (\nu_G^2 - 1)$$

$$c_G = \frac{N}{2} I_0 \Omega c_G^*$$

where  $I_0 = \int_0^R r^2 m dr$  and  $\nu_G$  is the natural frequency of the gimbal flap motion.

## Modal Equations

### Bending

Consider the equilibrium of the elastic, inertial, and centrifugal bending moments. From the above analysis, these terms give the homogeneous equation for bending of the blade:

$$\begin{aligned} & [(\epsilon I_{zz} \tau^2 + \epsilon I_{xx} \bar{k} \bar{k}) (\tau_0 \tau - x_0 \bar{k})''']'' \\ & - \Omega^2 \left[ \int_0^1 \rho m \bar{a}_y (\tau_0 \tau - x_0 \bar{k})'' \right]' \\ & - \bar{\Omega} \cdot \bar{\Omega} \cdot (\tau_0 \tau - x_0 \bar{k}) \\ & + m (\tau_0 \tau - x_0 \bar{k})'' = 0 \end{aligned}$$

This equation may be solved by the method of separation of variables.

Writing

$$(\tau_0 \tau - x_0 \bar{k}) = \bar{\eta}(r) e^{j\omega \tau}$$

it becomes

$$(\epsilon I \bar{\eta}''')'' - \Omega^2 \left[ \int_0^1 \rho m \bar{a}_y \bar{\eta}'' \right]' - \bar{\Omega} \cdot \bar{\Omega} \cdot \bar{\eta} - m \omega^2 \bar{\eta} = 0$$

This is the modal equation for coupled flap/lag bending of the rotating blade. It is an ordinary differential equation for the mode shape  $\bar{\eta}(r)$ : this mode may be interpreted as the free vibration of the rotating beam at natural frequency  $\omega$ .

This modal equation, with the appropriate boundary conditions for a cantilever or hinged blade, is a proper Sturm-Liouville eigenvalue problem. It follows that there exists a series of eigensolutions  $\bar{\eta}_k(r)$  of this equation, with corresponding eigenvalues  $\omega_k^2$ . The eigensolutions -- modes -- are orthogonal with weighting function  $m$ ; so if  $i \neq k$ ,

$$\int_0^1 \bar{\eta}_i \cdot \bar{\eta}_k m dr = 0$$

These modes form a complete series, so it is possible to expand the rotor blade bending as a series in the modes:

$$\tau_0 \tau - x_0 \bar{k} = \sum_{i=1}^{\infty} q_i(t) \bar{\eta}_i(r)$$

We shall normalize the bending modes to unit amplitude (nondimensional) at the tip:  $|\bar{y}_k(1)| = 1$ .

### Torsion

Consider the homogeneous equation for the elastic torsion motion of the nonrotating blade; i.e. the balance of structural and inertial torsion moments, which from the above analysis is:

$$-(GJ \Theta_c')' + I_\theta \ddot{\Theta}_c = 0$$

We could consider the equation for the torsion motion of the rotating blade, i.e. including centrifugal forces and some additional structural torsion moments. For the usual torsion stiffness of rotor blades these terms have little effect however, and the nonrotating torsion modes are an accurate representation of the blade motion. Solving this equation by separation of variables, write  $\Theta_c = \{(\tau)\} e^{i\omega\tau}$ , so:

$$-(GJ \{ \}' )' - I_\theta \omega^2 \{ \} = 0$$

This equation is a proper Sturm-Liouville eigenvalue problem, so it follows that there exists a series of eigensolutions  $\{k(\tau)\}$ , and corresponding eigenvalues  $\omega_k^2$  ( $k = 1 \dots \infty$ ). The modes are orthogonal with weighting function  $I_\theta$ , so if  $i \neq k$

$$\int_{r_{FA}} \{i\} \{k\} I_\theta dr = 0$$

The modes form a complete set, so the elastic torsion of the blade may be expanded as a series in the modes:

$$\Theta_c = \sum_{i=1}^{\infty} p_i(t) \{i(\tau)\}$$

These modes are the free vibration shape of the nonrotating blade in torsion, at natural frequency  $\omega_k$ . We shall normalize the torsion modes to unity at the tip,  $\{k(1) = 1$ .



### Expansion in modes

The bending and torsion motion of the blade is now expanded as series in the natural modes. By this means the partial differential equations for the motion (in  $r$  and  $t$ ) are converted to ordinary differential equations (in  $t$ ) for the degrees of freedom.

For the blade bending we write

$$(\vec{z}_0 \cdot \vec{r} - x_0 \vec{k}) = (\vec{z}_0 \cdot \vec{r} - x_0 \vec{k})_{\text{trim}} + \sum_{i=1}^{\infty} q_i(t) \vec{\eta}_i(r)$$

where  $\vec{\eta}_i$  are the rotating, coupled bending modes defined above. These modes are orthogonal, and satisfy the modal equation given above. The  $q_i$  are the degrees of freedom for the bending motion of the blade. It is assumed (for the inertial terms) that the trim bending deflection is steady, independent of time; and when the substitution for the modal expansion is made, the subscript "trim" will be dropped, as that is all that will be meant by  $(\vec{z}_0 \cdot \vec{r} - x_0 \vec{k})$  then.

For the blade elastic torsion we write

$$\theta_e = \sum_{i=1}^{\infty} p_i(t) \xi_i(r)$$

where  $\xi_i$  are the nonrotating elastic torsion modes. These modes are orthogonal, and satisfy the modal equation given above. The  $p_i$  ( $i \geq 1$ ) are the degrees of freedom for the elastic torsion motion of the blade.

We also have the rigid pitch degree of freedom

$$\rho_0 = \tilde{\theta}^0 = (\theta^0 - \theta^c) + \theta_{\text{con}}$$

which is the total rigid pitch motion of the blade. Since it is rigid pitch, rotation about the FA, it has mode shape  $\xi_0 \equiv 1$ . Thus the total pitch perturbation of the blade is expanded as a series:

$$\tilde{\theta} = \sum_{i=0}^{\infty} p_i(t) \xi_i(r)$$

For the blade pitch  $\Theta$  then, the mean plus the perturbation is

$$\Theta = \Theta_m + \tilde{\Theta} = (\Theta_{con} + \Theta_{tw}) + \sum_i p_i \xi_i$$

The subscript "m" on the trim pitch angle will be dropped when the substitution for the modal expansion is made, since that is all that will be meant by  $\Theta$  then.

#### Nonrotating Frame

The equations of motion and the hub forces and moments are in the rotating frame yet. To get to the nonrotating frame, we introduce a coordinate transformation of the Fourier type; i.e., introduce the new degrees of freedom:

$$\begin{aligned}\beta_0 &= \frac{1}{2} \sum_{n=1}^N q^{(m)} \\ \beta_{nc} &= \frac{2}{2} \sum_{n=1}^N q^{(m)} \cos n\psi_m \\ \beta_{ns} &= \frac{2}{2} \sum_{n=1}^N q^{(m)} \sin n\psi_m \\ \beta_{\frac{N}{2}} &= \frac{1}{2} \sum_{n=1}^N q^{(m)} (-1)^n\end{aligned}$$

where  $\beta_0$  is the coning mode;  $\beta_{nc}$  &  $\beta_{ns}$  the tip path plane tilt coordinates; and  $\beta_{\frac{N}{2}}$  is the reactionless flap mode -- for the out of plane bending of the blade. Then:

$$q^{(m)} = \beta_0 + \sum_n (\beta_{nc} \cos n\psi_m + \beta_{ns} \sin n\psi_m) + \beta_{\frac{N}{2}} (-1)^n$$

where the summation over n goes from 1 to (N-1)/2 for N odd; and from 1 to (N-2)/2 for N even; the  $\beta_{\frac{N}{2}}$  degree of freedom appears only if N is even.

The quantities  $\beta_0$ ,  $\beta_{nc}$ ,  $\beta_{ns}$ , and  $\beta_{\frac{N}{2}}$  are degrees of freedom, i.e. functions of time, just as the quantities  $q^{(m)}$  are. These degrees of freedom describe the rotor motion as seen in the nonrotating frame, while the  $q^{(m)}$  describe the motion in the rotating frame.

This coordinate transform must be accompanied by a conversion of the equations of motion for  $q^{(m)}$  from the rotating to the nonrotating frame. This is accomplished by operating on the equations of motion with the following summation operators:

$$\frac{1}{N} \sum_m (\dots), \quad \frac{2}{N} \sum_m (\dots) \cos n\psi_m, \quad \frac{2}{N} \sum_m (\dots) \sin n\psi_m, \quad \frac{1}{N} \sum_m (\dots) (-1)^m$$

Reference 4 gives more details of this transformation.

Similarly, the degrees of freedom for the blade pitch and gimbal motion are transformed to the nonrotating frame. The corresponding degrees of freedom for the rotating and nonrotating frames are:

<u>rotating</u>	<u>nonrotating</u>
$q_i^{(m)}$	$\beta_{0,1c,1s}^{(1)}$
$p_i^{(m)}$	$\theta_{0,1c,1s}^{(1)}$
$\beta_0, \theta_0, \psi_s$	$\beta_{0c}, \beta_{0s}, \psi_s$

When the transformation of the equations and degrees of freedom is accomplished, there is a decoupling of the inertial and structural terms as follows (for  $N \geq 3$ ):

- 0, 1C, 1S degrees of freedom;  $\beta_{0c}, \beta_{0s}$ , and  $\psi_s$ ; and the rotor shaft motion.
- 2C, 2S, ..., nc, ns,  $N/2$  degrees of freedom (as present).

The first set couples with the fixed system motion. The latter set is just internal rotor motion. For  $N = 3$ , the first set is the complete description of the motion of course. Nonaxial flow aerodynamics couples all the rotor degrees of freedom and shaft motion; i.e. the two sets above are coupled for helicopter forward flight or conversion mode operation. For axial flow -- hover or proprotor airplane mode cruise operation -- the aerodynamic terms decouple also.

We shall assume here that the rotor has three or more blades,  $N \geq 3$ . For  $N = 2$ , there are periodic coefficients even in the inertia terms, so

that is a special case. For the case of periodic coefficients in the aerodynamics, i.e. helicopter forward flight or conversion mode flight, it is necessary to specify  $N$ ; we shall take  $N = 3$  for that case. (The periodic coefficients depend on  $N$ .) For the case of axial flow, or for the constant coefficient approximation for the nonaxial flow case, the equations obtained will be valid for all  $N$  greater than or equal to 3.

Reference 4 discusses these points further.

#### Equations of Motion/ Hub Forces and Moments

The elements are available now to obtain the equations of motion for the blade bending and torsion modes, in the rotating frame; and the forces and moments acting on the hub due to the  $m$ th blade. The steps required are:

- a) Substitute for the expansion of the bending and torsion motion as a series in the modes.
- b) Use the appropriate modal equation to introduce the mode natural frequency into the bending or torsion equation, replacing the structural stiffness terms (and for bending also some of the centrifugal stiffness terms).
- c) For the bending equation, operate with  $\int_0^1 \vec{r}_k \cdot (\dots) dr$  to obtain the ordinary differential equation for the  $k$ th mode of the  $m$ th blade (the  $q_k$  equation).
- d) For the torsion equation, operate with  $\int_{CPA} \hat{z}_k (\dots) dr$  to obtain the ordinary differential equation for the  $k$ th mode of the  $m$ th blade (the  $p_k$  equation).

The result is the equations of motion and hub forces in the rotating frame. The transformation to the nonrotating frame involves the following steps:

- a) Operate on the hub force and moment with  $\sum (\dots)$  ;  
i.e. sum over all N blades to obtain the total force and moment on the hub.
- b) Find the  $\tau_s, \gamma_s, \kappa_s$  components of the force and moment in the nonrotating frame (the S system).
- c) Write the shaft motion  $\vec{a}_0, \vec{\omega}_0$ , and  $\vec{\dot{\omega}}_0$  in terms of the  $\tau_s, \gamma_s, \kappa_s$  components in the nonrotating frame (the S system).
- d) Apply the Fourier coordinate transform to the equations of motion and rotor degrees of freedom; operate on the equations for bending and torsion with

$$\frac{1}{N} \sum (\dots), \frac{2}{N} \sum (\dots) \cos \psi_m, \frac{2}{N} \sum (\dots) \sin \psi_m, \dots$$

to obtain the nonrotating equations of motion (0, 1C, 1S, etc.).  $N \geq 3$  is assumed for this transformation.

The transformation of the equations to the nonrotating frame will be delayed however, so the rotating modal equations may be presented first.

We add at this point structural damping terms, modelled as equivalent viscous damping; the structural damping parameter is  $g_s$  (which may be different for each degree of freedom), equal to twice the equivalent damping ratio.

Names are given to all the inertial constants now. The equations of motion, hub forces and moments, and inertia constants are also normalized at this point. The inertia constants are divided by the characteristic inertia  $I_b = \int_0^R r^2 m dr$ , and we introduce the blade Lock number  $\delta = \frac{1}{2} a c R^4 / I_b$ . This normalization of the inertia constants is denoted by a superscript \*. The rotating equations of motion are divided by  $I_b$ ; the hub forces and moments are divided by  $(N/2)I_b$  for  $M_x, M_y, H$ , and  $Y$ , and by  $NI_b$  for  $Q$  and  $T$ . The result is that the forces and moments are obtained in coefficient form. More details of this normalization procedure are given in reference 4.

### Equations

The resulting hub forces, hub moments, gimbal equations, and equations of motion for coupled flap/lag bending and for elastic torsion/rigid pitch of the rotating blade are as follows.

Forces:

$$\begin{aligned}\delta \frac{2C_H}{\sigma_a} &= \delta \left( \frac{2C_H}{\sigma_a} \right)_{aero} - 2M_b^* \ddot{y}_h + \sum S_{q_i}^* \cdot \vec{L}_B \ddot{\beta}_{1s}^{(i)} \\ \delta \frac{2C_V}{\sigma_a} &= \delta \left( \frac{2C_V}{\sigma_a} \right)_{aero} - 2M_b^* \ddot{y}_h - \sum S_{q_i}^* \cdot \vec{L}_B \ddot{\beta}_{1c}^{(i)} \\ \delta \frac{C_T}{\sigma_a} &= \delta \left( \frac{C_T}{\sigma_a} \right)_{aero} - M_b^* \ddot{z}_h - \sum S_{q_i}^* \cdot \vec{L}_B \ddot{\beta}_{1o}^{(i)}\end{aligned}$$

Moments:

$$\begin{aligned}\delta \frac{2C_{M_x}}{\sigma_a} &= \delta \left( \frac{2C_{M_x}}{\sigma_a} \right)_{aero} - I_o^* (\ddot{\alpha}_x + 2\dot{\alpha}_y) - I_o^* (\ddot{\beta}_{6s} - 2\dot{\beta}_{6c}) \\ &\quad - \sum I_{q_i \alpha}^* \cdot \vec{L}_B (\ddot{\beta}_{1s}^{(i)} - 2\dot{\beta}_{1c}^{(i)}) \\ &\quad + \sum S_{p_i \alpha}^* \cdot \vec{L}_B (\ddot{\theta}_{1s}^{(i)} - 2\dot{\theta}_{1c}^{(i)}) \\ &\quad - 2 \sum I_{q_i \alpha}^* (\dot{\beta}_{1s}^{(i)} - \beta_{1c}^{(i)})\end{aligned}$$

$$\begin{aligned}\delta \frac{2C_{M_y}}{\sigma_a} &= \delta \left( \frac{2C_{M_y}}{\sigma_a} \right)_{aero} - I_o^* (\ddot{\alpha}_y - 2\dot{\alpha}_x) + I_o^* (\ddot{\beta}_{6c} + 2\dot{\beta}_{6s}) \\ &\quad + \sum I_{q_i \alpha}^* \cdot \vec{L}_B (\ddot{\beta}_{1c}^{(i)} + 2\dot{\beta}_{1s}^{(i)}) \\ &\quad - \sum S_{p_i \alpha}^* \cdot \vec{L}_B (\ddot{\theta}_{1c}^{(i)} + 2\dot{\theta}_{1s}^{(i)}) \\ &\quad + 2 \sum I_{q_i \alpha}^* (\dot{\beta}_{1c}^{(i)} + \beta_{1s}^{(i)})\end{aligned}$$

$$\begin{aligned}\delta \frac{C_\theta}{\sigma_a} &= \delta \left( \frac{C_\theta}{\sigma_a} \right)_{aero} + I_o^* \ddot{\alpha}_z + I_o^* \ddot{\psi}_s \\ &\quad + \sum I_{q_i \alpha}^* \cdot \vec{L}_B \ddot{\beta}_{1o}^{(i)} - \sum S_{p_i \alpha}^* \cdot \vec{L}_B \ddot{\theta}_{1o}^{(i)} \\ &\quad - 2 \sum I_{q_i \alpha}^* \dot{\beta}_{1o}^{(i)}\end{aligned}$$

Gimbal:

$$\begin{aligned}\delta \frac{2C_{M_x}}{\sigma_a} + I_o^* C_\theta^* \dot{\beta}_{6c} + I_o^* (v_\theta^2 - 1) \beta_{6c} &= 0 \\ -\delta \frac{2C_{M_y}}{\sigma_a} + I_o^* C_\theta^* \dot{\beta}_{6s} + I_o^* (v_\theta^2 - 1) \beta_{6s} &= 0\end{aligned}$$

Bending:

$$\begin{aligned}
 & I_{qk}^* (\ddot{q}_k + g_s v_k \dot{q}_k + v_k^2 q_k) + 2 \sum I_{qkq_i}^* \dot{q}_i \\
 & - \sum S_{qkp_i}^* \ddot{p}_i - \sum S_{qkp_i}^* \dot{p}_i \\
 & + I_{qk\alpha}^* \cdot \vec{L}_B \ddot{\psi}_s + I_{qk\alpha}^* \cdot \vec{L}_B (\ddot{\beta}_G + \beta_G) \\
 & + 2 I_{qk\psi}^* \dot{\psi}_s - I_{qk\alpha}^* (\ddot{\theta}_G - \theta_G + 2\dot{\beta}_G) \\
 & + S_{qk}^* \cdot \vec{L}_B \ddot{z}_n - S_{qk}^* \cdot \vec{L}_B (\ddot{x}_n \sin \psi_n - \ddot{y}_n \cos \psi_n) \\
 & + I_{qk\alpha}^* \cdot \vec{L}_B \ddot{\alpha}_2 + I_{qk\alpha}^* \cdot \vec{L}_B ((\ddot{\alpha}_x + 2\dot{\alpha}_y) \sin \psi_n \\
 & \quad - (\ddot{\alpha}_y - 2\dot{\alpha}_x) \cos \psi_n) \\
 & = \delta \frac{M_{qk} a_{el0}}{ac} + I_{qk0}^*
 \end{aligned}$$

Torsion/pitch:

$$\begin{aligned}
 & I_{pk}^* (\ddot{p}_k + g_s \omega_k \dot{p}_k + \omega_k^2 p_k) \\
 & + \sum I_{pkp_i}^* \ddot{p}_i + \sum I_{pkp_i}^* \dot{p}_i \\
 & - \sum S_{pkq_i}^* \ddot{q}_i - \sum S_{pkq_i}^* \dot{q}_i \\
 & + I_{pk\alpha}^* \cdot \vec{L}_B \ddot{\psi}_s - I_{pk\alpha}^* \cdot \vec{L}_B (\ddot{\beta}_G + \beta_G) \\
 & - S_{pk\alpha}^* (\ddot{\theta}_G + 2\dot{\beta}_G - \theta_G) + S_{pk\alpha}^* \beta_G \\
 & - S_{pk}^* \cdot \vec{L}_B \ddot{z}_n - S_{pk}^* \cdot \vec{L}_B (\ddot{x}_n \sin \psi_n - \ddot{y}_n \cos \psi_n) \\
 & + I_{pk\alpha}^* \cdot \vec{L}_B \ddot{\alpha}_2 - I_{pk\alpha}^* \cdot \vec{L}_B ((\ddot{\alpha}_x + 2\dot{\alpha}_y) \sin \psi_n \\
 & \quad - (\ddot{\alpha}_y - 2\dot{\alpha}_x) \cos \psi_n) \\
 & = \delta \frac{M_{pk} a_{el0}}{ac} + (I_{pk0}^* \omega_0^2 j_k(r_{pk})) \theta_{el0}
 \end{aligned}$$

Aerodynamic forces

$$\frac{2C_H}{\pi} = \frac{2}{\pi} \sum_m \left[ \sin \psi_m \int_0^1 \frac{F_x}{a c} dr + \cos \psi_m \int_0^1 \frac{F_r}{a c} - \frac{F_z}{a c} (\beta_0 + \delta_1 - \delta_2 + K_B \cdot (x_0 r^2 + z_0 r^2))' \right] dr$$

$$- \frac{2C_V}{\pi} = \frac{2}{\pi} \sum_m \left[ \cos \psi_m \int_0^1 \frac{F_x}{a c} dr - \sin \psi_m \int_0^1 \frac{F_r}{a c} - \frac{F_z}{a c} (\beta_0 + \delta_1 - \delta_2 + K_B \cdot (x_0 r^2 + z_0 r^2))' \right] dr$$

$$\frac{C_T}{\pi} = \frac{1}{2} \sum_m \int_0^1 \frac{F_z}{a c} dr$$

$$\frac{2C_{M0}}{\pi} = \frac{2}{2} \sum_m \sin \psi_m \int_0^1 \frac{F_z}{a c} r dr$$

$$- \frac{2C_{M1}}{\pi} = \frac{2}{2} \sum_m \cos \psi_m \int_0^1 \frac{F_z}{a c} r dr$$

$$\frac{C_D}{\pi} = \frac{1}{2} \sum_m \int_0^1 \frac{F_x}{a c} r dr$$

$$\frac{M_{qk}^{aero}}{a c} = \int_0^1 \gamma_k \cdot \left( \frac{F_z}{a c} z_0 - \frac{F_x}{a c} K_B \right) dr$$

$$\frac{M_{pk}^{aero}}{a c} = \int_{r_{FA}}^1 \beta_k M_0 dr - \int_{r_{FA}}^1 \left( \frac{F_x}{a c} z_0 + \frac{F_z}{a c} K_B \right) \cdot \tilde{X}_{Ak} dr$$

where

$$\tilde{X}_{Ak} = \tilde{X}_k - \beta_k x_{zk}$$

and then for the nonrotating equations:

$$M_{p0}^{aero} = \frac{1}{2} \sum_m M_{qk}^{aero}$$

$$M_{p1}^{aero} = \frac{2}{2} \sum_m M_{qk}^{aero} \cos \psi_m$$

$$M_{p15}^{aero} = \frac{2}{2} \sum_m M_{qk}^{aero} \sin \psi_m$$

$$M_{\theta 0}^{aero} = \frac{1}{2} \sum_m M_{pk}^{aero}$$

$$M_{\theta 1}^{aero} = \frac{2}{2} \sum_m M_{pk}^{aero} \cos \psi_m$$

$$M_{\theta 15}^{aero} = \frac{2}{2} \sum_m M_{pk}^{aero} \sin \psi_m$$



Inertia constants

$$M_b^* = \int_0^1 m dr / I_b$$

$$S_{q_i}^* = \int_0^1 \vec{\eta}_i m dr / I_b$$

$$I_0^* = \int_0^1 r^2 m dr / I_b$$

$$I_{q_i \alpha}^* = \int_0^1 \vec{\eta}_i r m dr / I_b$$

$$S_{r_i \alpha}^* = \frac{1}{I_b} \int_{r_{FA}}^1 \left( \xi_i (x_0 \vec{r} + z_0 \vec{k} + x_z \vec{r}) + \xi_i (r_{FA}) (\delta_3 \vec{r}_B - \delta_2 \vec{k}_B) (r - r_{FA}) \right) r m dr$$

$$I_{\dot{q}_i \alpha}^* = \frac{1}{I_b} \int_0^1 \vec{k}_B \cdot \vec{\eta}_i \left( -z_{FA} + r \delta_1 - (r - r_{FA}) \delta_2 + \vec{k}_B \cdot (z_0 \vec{r} - x_0 \vec{k} - x_z \vec{r}) \right) m dr$$

$$I_{\dot{q}_i \psi}^* = \frac{1}{I_b} \left[ \int_0^1 r m \int_0^r \vec{\eta}_i' \cdot (z_0 \vec{r} - x_0 \vec{k} - x_z \vec{r})' dy dr \right. \\ \left. - \int_0^1 \vec{k}_B \cdot \vec{\eta}_i (-r \delta_1 + (r - r_{FA}) \delta_2) m dr \right. \\ \left. - \int_0^1 \vec{k}_B \cdot \vec{\eta}_i (x_{FA} + r_{FA} \delta_3 + \vec{k}_B \cdot (z_0 \vec{r} - x_0 \vec{k} - x_z \vec{r})) m dr \right]$$

$$I_0^* (\psi_0^2 - 1) = k_G / \left( \frac{N}{2} I_b \Omega^2 \right)$$

$$I_0^* (c_G^*) = c_G / \left( \frac{N}{2} I_b \Omega^2 \right)$$

$$I_{q_k}^* = \int_0^1 \gamma_k^2 m dr / I_b$$

$$S_{q_k \ddot{\rho}_i}^* = \frac{1}{I_b} \int_{r_{FA}}^1 \vec{\eta}_k \cdot \left( \xi_i (x_0 \vec{r} + z_0 \vec{k} + x_z \vec{r}) - \xi_i (r_{FA}) (\delta_2 \vec{k}_B - \delta_3 \vec{r}_B) (r - r_{FA}) \right) m dr$$

$$S_{qkpi}^* = \frac{1}{\pm_b} \left[ \int_{r_{FA}}^1 \vec{\gamma}_k \cdot \left\{ - \left[ \left( \vec{z}_i \cdot (x_0 \vec{r} + z_0 \vec{k} + x_z \vec{r}) \right)^\vee \int_r^1 g m d\varphi \right]^\vee \right. \right. \\ \left. \left. + [ (x_c - x_z) \vec{z}_i \cdot \vec{r} m ]^\vee \right. \right. \\ \left. \left. - m \vec{k}_B \cdot \vec{z}_i \cdot \vec{k}_B \cdot (x_0 \vec{r} + z_0 \vec{k} + x_z \vec{r}) \right. \right. \\ \left. \left. + \vec{z}_i \cdot (r_{FA}) m (r \delta_3 \vec{r}_B - r_{FA} \delta_2 \vec{k}_B) \right\} d\tau \right. \\ \left. - \vec{\gamma}_k(r_{FA}) \cdot (\delta_3 \vec{r}_B - \delta_2 \vec{k}_B) \cdot \vec{z}_i \cdot (r_{FA}) \int_{r_{FA}}^1 r m d\tau \right. \\ \left. - \int_{r_{FA}}^1 \vec{\gamma}_k \cdot \left[ \left( \frac{E \vec{x}^p}{\Sigma^2} \vec{k} - \frac{E \vec{x}^p}{\Sigma^2} \vec{r} \right) \Theta_m^\vee \vec{z}_i^\vee \right]^\vee d\tau \right]$$

$$I_{qkq_i}^* = \frac{1}{\pm_b} \left[ - \int_0^1 \vec{\gamma}_k \cdot \vec{k}_B m \int_0^r \vec{\gamma}_i^\vee \cdot (z_0 \vec{r} - x_0 \vec{k} - x_z \vec{k})^\vee d\varphi d\tau \right. \\ \left. + \int_0^1 \vec{\gamma}_i \cdot \vec{k}_B m \int_0^r \vec{\gamma}_k^\vee \cdot (z_0 \vec{r} - x_0 \vec{k} - x_c \vec{k})^\vee d\varphi d\tau \right. \\ \left. - \int_0^1 \vec{\gamma}_k \cdot \left[ ((x_c - x_z) \vec{k} m \vec{k}_B \cdot \vec{\gamma}_i)^\vee \right. \right. \\ \left. \left. - m (\delta_1 - \delta_2) \vec{z}_i \cdot \vec{\gamma}_i \right] d\tau \right. \\ \left. + \vec{\gamma}_k(r_{FA}) \cdot (\delta_2 \vec{r}_B + \delta_3 \vec{k}_B) \int_{r_{FA}}^1 \vec{k}_B \cdot \vec{\gamma}_i m d\tau \right]$$

$$I_{qk\psi}^* = \frac{1}{\pm_b} \left[ \int_0^1 r m \int_0^r \vec{\gamma}_k^\vee \cdot (z_0 \vec{r} - x_0 \vec{k} - x_c \vec{k})^\vee d\varphi d\tau \right. \\ \left. - \int_0^1 \vec{\gamma}_k \cdot \vec{k}_B [x_{FA} + r_{FA} \delta_3 + \vec{k}_B \cdot (z_0 \vec{r} - x_0 \vec{k} - x_z \vec{k})] m d\tau \right. \\ \left. - \int_0^1 \vec{\gamma}_k \cdot \left[ ((x_c - x_z) \vec{k} r m)^\vee - m (\delta_1 - \delta_2) \vec{r}_B \cdot \vec{r} \right] d\tau \right. \\ \left. + \vec{\gamma}_k(r_{FA}) \cdot (\delta_2 \vec{r}_B + \delta_3 \vec{k}_B) \int_{r_{FA}}^1 r m d\tau \right]$$

$$I_{qk0}^* = \frac{1}{\pm_b} \left[ - \int_0^1 \vec{\gamma}_k \cdot \left\{ ((x_c \vec{k})^\vee \int_r^1 g m d\varphi)^\vee + ((x_z - x_c) \vec{k} m r)^\vee \right\} d\tau \right. \\ \left. + m \vec{r}_B \cdot (\delta_1 - \delta_2) - m \vec{k}_B \cdot (x_{FA} + r_{FA} \delta_3 - x_z \cos \theta) \right] d\tau \\ \left[ - \vec{\gamma}_k(r_{FA}) \cdot (\delta_2 \vec{r}_B + \delta_3 \vec{k}_B) \int_{r_{FA}}^1 r m d\tau \right]$$

and with

$$\vec{X}_k = \xi_k \vec{r} - \int_{r_{FA}}^r \xi_k (z_0 \vec{r} - x_0 \vec{r})' (r-g) dg$$

$$I_{PK}^* = \int_{r_{FA}}^1 \xi_k^2 I_0 dr / I_b$$

$$S_{PK}^* = \int_{r_{FA}}^1 \vec{X}_k m dr / I_b$$

$$I_{PK\alpha}^* = \int_{r_{FA}}^1 \vec{X}_k r m dr / I_b$$

$$S_{PK\ddot{\alpha}}^* = \pm_b \int_{r_{FA}}^1 \vec{X}_k \cdot \vec{r}_B [-z_{FA} + r\delta_1 - (r-r_{FA})\delta_2 + \vec{r}_B \cdot (z_0 \vec{r} - x_0 \vec{r} - x_z \vec{r})] m dr$$

$$S_{PK\ddot{\eta}_i}^* = \pm_b \int_{r_{FA}}^1 \vec{X}_k \cdot (-\vec{J} \times \vec{\eta}_i) m dr$$

$$I_{PK\ddot{r}_i}^* = \pm_b \left[ -\int_{r_{FA}}^1 \xi_i [\vec{X}_k \cdot (z_0 \vec{r} - x_0 \vec{r} - x_z \vec{r}) + \xi_k x_z^2] m dr + \int_{r_{FA}}^1 \vec{X}_k \cdot (\delta_2 \vec{r}_B + \delta_3 \vec{r}_B) (r-r_{FA}) \xi_i(r_{FA}) m dr + \int_{r_{FA}}^1 [(\xi_k - \xi_k(r_{FA})) \xi_i(r_{FA}) + (\xi_i - \xi_i(r_{FA})) \xi_k(r_{FA})] I_0 dr \right]$$

$$I_{PK\rho_i}^* = \pm_b \left[ \int_{r_{FA}}^1 \xi_k \xi_i I_0 (\cos^2 \theta - \sin^2 \theta) dr - \int_{r_{FA}}^1 (\xi_k - \xi_k(r_{FA})) \left[ (k_p^2 \int_{r_{FA}}^1 g m dg + \theta_{Tw}^2 \frac{E I_{PP}}{S Z^2}) \xi_i' \right]' dr \right]$$

$$S_{PK\alpha}^* = k_{PG} I_{P0}^* \omega_0^2 \xi_k(r_{FA})$$

$$S_{pq_i}^* = \frac{1}{\mathbb{I}_B} \left[ \int_{r_{FA}}^1 \vec{\chi}_k \cdot \vec{k}_B \vec{r}_B \cdot \vec{\eta}_i \, m \, dr \right. \\
+ \int_{r_{FA}}^1 \beta_k k_z \, m \, r \cdot (\vec{\eta}_i^* r - \vec{\eta}_i) \, dr \\
+ \int_{r_{FA}}^1 \beta_k (x_0 \vec{r} + z_0 \vec{k})^m \cdot \int_r^1 (r \vec{\eta}_i - g \vec{\eta}_i(r)) \, m \, dg \, dr \\
- \int_{r_{FA}}^1 \beta_k \vec{\eta}_i^* \cdot \int_r^1 \left\{ r \vec{r}_B \vec{r}_B \cdot (x_0 \vec{r} + z_0 \vec{k} + x_z \vec{r}) \right. \\
+ g \vec{k}_B \vec{k}_B \cdot (x_0 \vec{r} + z_0 \vec{k} + x_z \vec{r}) \\
- g (x_0 \vec{r} + z_0 \vec{k}) |_r \\
+ (g-r) \vec{r}_B (x_{FA} + r_{FA} \delta_3) \\
\left. + (g-r) \vec{k}_B g (\delta_1 - \delta_2) \right\} \, m \, dg \, dr \\
\left. + \int_{r_{FA}}^1 \beta_k \left[ \Theta_{TW}^* \left( \frac{E I_{kT} r}{S^2 z^2} \vec{k} - \frac{E I_{kT} r}{S^2 z^2} \vec{r} \right) \cdot \vec{\eta}_i^* \right]^* \, dr \right]$$

except for rigid pitch ( $k = 0$ ), where

$$\vec{\chi}_0 = -(z_0 \vec{r} - x_0 \vec{k} - x_z \vec{k}) + (\delta_2 \vec{r}_B + \delta_3 \vec{k}_B)(r - r_{FA}) \\
+ (z_0 \vec{r} - x_0 \vec{k}) |_{r_{FA}} \\
+ (z_0 \vec{r} - x_0 \vec{k})^* |_{r_{FA}} (r - r_{FA})$$

and:

$$\begin{aligned}
 S_{p0i}^* = \frac{1}{I_b} & \left[ \int_{r_{FA}}^1 \vec{x}_0 \cdot \vec{k}_B \vec{t}_B \cdot \vec{\eta}_i \, m \, dr \right. \\
 & + \int_{r_{FA}}^1 (\vec{\eta}_i - r \vec{\eta}_i^{\vee}(r_{FA})) \cdot \left\{ \begin{aligned} & -\vec{E}_B \vec{E}_B \cdot (x_0 \vec{t} + z_0 \vec{k} + x_z \vec{t}) \\ & + (x_0 \vec{t} + z_0 \vec{k})|_{r_{FA}} \\ & - r_{FA} (x_0 \vec{t} + z_0 \vec{k})^{\vee}|_{r_{FA}} \\ & - \vec{t}_B (x_{FA} + r_{FA} \delta_3) \\ & - \vec{E}_B r (\delta_1 - \delta_2) \\ & + r_{FA} (\delta_3 \vec{t}_B - \delta_2 \vec{E}_B) \end{aligned} \right\} m \, dr \\
 & + \int_{r_{FA}}^1 (\vec{\eta}_i - r \vec{\eta}_i^{\vee})|_{r_{FA}} \cdot \left\{ \begin{aligned} & -\vec{t}_B \vec{t}_B \cdot (x_0 \vec{t} + z_0 \vec{k} + x_z \vec{t}) \\ & + r (x_0 \vec{t} + z_0 \vec{k})^{\vee}|_{r_{FA}} \\ & + \vec{t}_B (x_{FA} + r_{FA} \delta_3) \\ & + \vec{E}_B r (\delta_1 - \delta_2) \end{aligned} \right\} m \, dr \\
 & + \vec{\eta}_i^{\vee}(r_{FA}) \cdot (\delta_3 \vec{t}_B - \delta_2 \vec{E}_B) \int_{r_{FA}}^1 r^2 m \, dr \\
 & - K_{Pi} \omega_0^2 I_{p0}
 \end{aligned}$$

### Nonrotating frame equations

The equations of motion for the rotor in the nonrotating frame, i.e. after application of the Fourier coordinate transformation, are

$$A_2 \ddot{x}_R + A_1 \dot{x}_R + A_0 x_R + \tilde{A}_2 \ddot{\alpha} + \tilde{A}_1 \dot{\alpha} + \tilde{A}_0 \alpha = B v_R + M_{aero}$$

and the hub forces and moments

$$F = C_2 \ddot{x}_R + C_1 \dot{x}_R + C_0 x_R + \tilde{C}_2 \ddot{\alpha} + \tilde{C}_1 \dot{\alpha} + \tilde{C}_0 \alpha + F_{aero}$$

where the rotor degrees of freedom ( $\vec{x}_R$ ), shaft motion ( $\vec{q}$ ), rotor blade pitch input ( $\vec{v}_R$ ), and the hub forces and moments ( $\vec{F}$ ) are:

$$\vec{x}_R = \begin{bmatrix} \beta_0^{(k)} \\ \beta_{1c}^{(k)} \\ \beta_{1s}^{(k)} \\ \theta_0^{(k)} \\ \theta_{1c}^{(k)} \\ \theta_{1s}^{(k)} \\ \beta_{GC} \\ \beta_{GS} \\ w_s \end{bmatrix} \quad \vec{q} = \begin{bmatrix} x_h \\ y_h \\ z_h \\ \phi_x \\ \phi_y \\ \phi_z \end{bmatrix}$$

$$\vec{v}_R = \begin{bmatrix} \dot{\theta}_0^{com} \\ \dot{\theta}_{1c}^{com} \\ \dot{\theta}_{1s}^{com} \end{bmatrix} \quad \vec{M} = \gamma \begin{bmatrix} \frac{\partial F}{\partial \beta_0} \\ \frac{\partial F}{\partial \beta_{1c}} \\ \frac{\partial F}{\partial \beta_{1s}} \\ \frac{\partial F}{\partial \theta_0} \\ \frac{\partial F}{\partial \theta_{1c}} \\ \frac{\partial F}{\partial \theta_{1s}} \\ \frac{\partial F}{\partial \beta_{GC}} \\ \frac{\partial F}{\partial \beta_{GS}} \\ \frac{\partial F}{\partial w_s} \end{bmatrix}$$

The matrices of the coefficients, and the aerodynamic forcing vectors, follow.

$$A_2 =$$

$H_{q_k}^*$			$-S_{q_k p_i}^*$					$H_{q_k \alpha}^* \cdot \vec{L}_B$
	$H_{q_k}^*$			$-S_{q_k p_i}^*$		$H_{q_k \alpha}^* \cdot \vec{L}_B$	$-H_{q_k \alpha}^*$	
		$H_{q_k}^*$			$-S_{q_k p_i}^*$	$H_{q_k \alpha}^*$	$H_{q_k \alpha}^* \cdot \vec{L}_B$	
$-S_{p_k q_i}^*$			$H_{p_k}^* + H_{p_k p_i}^*$					$H_{p_k \alpha}^* \cdot \vec{L}_B$
	$-S_{p_k q_i}^*$			$H_{p_k}^* + H_{p_k p_i}^*$		$-H_{p_k \alpha}^* \cdot \vec{L}_B$	$-S_{p_k \alpha}^*$	
		$-S_{p_k q_i}^*$			$H_{p_k}^* + H_{p_k p_i}^*$	$S_{p_k \alpha}^*$	$-H_{p_k \alpha}^* \cdot \vec{L}_B$	
	$H_{q_i \alpha}^* \cdot \vec{L}_B$			$-S_{p_i \alpha}^* \cdot \vec{L}_B$		$H_0^*$		
		$H_{q_i \alpha}^* \cdot \vec{L}_B$			$-S_{p_i \alpha}^* \cdot \vec{L}_B$		$H_0^*$	

$$A_1 =$$

$\begin{matrix} \mathcal{I}_{q_k}^* g_s v_k \\ + 2 \mathcal{I}_{q_k}^* \dot{q}_i \end{matrix}$							$2 \mathcal{I}_{q_k}^* \dot{\psi}$
$\begin{matrix} \mathcal{I}_{q_k}^* g_s v_k \\ + 2 \mathcal{I}_{q_k}^* \dot{q}_i \end{matrix}$	$2 \mathcal{I}_{q_k}^*$			$-2 \mathcal{S}_{q_k \ddot{p}_i}^*$		$2 \mathcal{I}_{q_k \dot{\alpha} i}^* \vec{l}_B$	
$-2 \mathcal{I}_{q_k}^*$	$\begin{matrix} \mathcal{I}_{q_k}^* g_s v_k \\ + 2 \mathcal{I}_{q_k}^* \dot{q}_i \end{matrix}$		$2 \mathcal{S}_{q_k \ddot{p}_i}^*$		$-2 \mathcal{I}_{q_k \dot{\alpha} i}^* \vec{l}_B$		
		$\mathcal{I}_{p_k}^* g_s u_k$					
	$-2 \mathcal{S}_{p_k \ddot{q}_i}^*$		$\mathcal{I}_{p_k}^* g_s u_k$	$2 \mathcal{I}_{p_k}^*$ $+ 2 \mathcal{I}_{p_k}^* \ddot{p}_i$		$-2 \mathcal{I}_{p_k \dot{\alpha} i}^* \vec{l}_B$	
$2 \mathcal{S}_{p_k \ddot{q}_i}^*$			$-2 \mathcal{I}_{p_k}^*$ $-2 \mathcal{I}_{p_k}^* \ddot{p}_i$	$\mathcal{I}_{p_k}^* g_s u_k$	$2 \mathcal{I}_{p_k \dot{\alpha} i}^* \vec{l}_B$		
$2 \mathcal{I}_{q_i \alpha}^*$	$2 \mathcal{I}_{q_i \alpha}^* \vec{l}_B$				$\mathcal{I}_0^* \vec{c}_G^*$	$2 \mathcal{I}_0^*$	
$-2 \mathcal{I}_{q_i \alpha}^* \vec{l}_B$	$2 \mathcal{I}_{q_i \alpha}^*$		$2 \mathcal{S}_{p_i \dot{\alpha} i}^* \vec{l}_B$		$-2 \mathcal{I}_0^*$	$\mathcal{I}_0^* \vec{c}_G^*$	



$$A_0 =$$

$I_{q_k}^* \dot{\varphi}_k^2$			$-S_{q_k p_i}^*$					
	$I_{q_k}^* (\dot{\varphi}_k^2 - 1)$	$I_{q_k}^* \dot{q}_s \dot{\varphi}_k + 2I_{q_k}^* \dot{q}_i$		$S_{q_k \ddot{p}_i}^*$ $-S_{q_k p_i}^*$				
	$-I_{q_k}^* \dot{q}_s \dot{\varphi}_k$ $-2I_{q_k}^* \dot{q}_i$	$I_{q_k}^* (\dot{\varphi}_k^2 - 1)$		$S_{q_k \ddot{p}_i}^*$ $-S_{q_k p_i}^*$				
$-S_{p_k q_i}^*$			$I_{p_k}^* \dot{\omega}_k^2 + I_{p_k}^* \dot{p}_i$					
	$S_{p_k \ddot{q}_i}^*$ $-S_{p_k q_i}^*$		$I_{p_k}^* (\dot{\omega}_k^2 - 1) + I_{p_k}^* \dot{p}_i - I_{p_k}^* \ddot{p}_i$	$I_{p_k}^* \dot{q}_s \dot{\omega}_k$	$S_{p_k \ddot{q}_i}^*$			
		$S_{p_k \ddot{q}_i}^*$ $-S_{p_k q_i}^*$		$-I_{p_k}^* \dot{q}_s \dot{\omega}_k + I_{p_k}^* \dot{p}_i - I_{p_k}^* \ddot{p}_i$		$S_{p_k \ddot{q}_i}^*$		
		$2I_{q_i}^* \dot{\alpha}$				$I_0^* (\dot{\varphi}_0^2 - 1)$		
	$-2I_{q_i}^* \dot{\alpha}$						$I_0^* (\dot{\varphi}_0^2 - 1)$	

$$\tilde{A}_2 =$$

		$S_{qk}^* \cdot \vec{L}_B$		$I_{qk\alpha}^* \cdot \vec{k}_B$	
	$S_{qk}^* \cdot \vec{k}_B$			$-I_{qk\alpha}^* \cdot \vec{L}_B$	
$-S_{qk}^* \cdot \vec{k}_B$			$I_{qk\alpha}^* \cdot \vec{L}_B$		
		$-S_{pk}^* \cdot \vec{k}_B$		$I_{pk\alpha}^* \cdot \vec{L}_B$	
	$S_{pk}^* \cdot \vec{L}_B$			$I_{pk\alpha}^* \cdot \vec{k}_B$	
$-S_{pk}^* \cdot \vec{L}_B$			$-I_{pk\alpha}^* \cdot \vec{k}_B$		
				$-I_0^*$	
			$I_0^*$		

$$\tilde{A}_1 =$$

			$2I_{q\alpha}^* \cdot \vec{l}_B$		
				$2I_{q\alpha}^* \cdot \vec{l}_B$	
			$-2I_{p\alpha}^* \cdot \vec{k}_B$		
				$-2I_{p\alpha}^* \cdot \vec{k}_B$	
			$2I_0^*$		
				$2I_0^*$	

B =

$\mathcal{I}_{p_0}^* \omega_0^2 \mathcal{J}_K(r_A)$		
	$\mathcal{I}_{p_0}^* \omega_0^2 \mathcal{J}_K(r_A)$	
		$\mathcal{I}_{p_0}^* \omega_0^2 \mathcal{J}_K(r_A)$

$$C_2 =$$

$-S_{q,i}^* \cdot \vec{L}_B$							
		$S_{q,i}^* \cdot \vec{L}_B$					
	$S_{q,i}^* \cdot \vec{L}_B$						
$I_{q,\alpha}^* \cdot \vec{L}_B$			$-S_{p,\alpha}^* \cdot \vec{L}_B$				$H_o^*$
	$I_{q,\alpha}^* \cdot \vec{L}_B$			$-S_{p,\alpha}^* \cdot \vec{L}_B$		$H_o^*$	
		$I_{q,\alpha}^* \cdot \vec{L}_B$			$-S_{p,\alpha}^* \cdot \vec{L}_B$		$H_o^*$

$C_1 =$

$-2I_{q_i}^*$							
	$2I_{q_i}^*$	$2I_{q_i}^* \vec{l}_B$		$-2S_{p,i}^* \vec{l}_B$		$2I_o^*$	
	$-2I_{q_i}^* \vec{l}_B$	$2I_{q_i}^*$		$2S_{p,i}^* \vec{l}_B$		$-2I_o^*$	

$$C_0 =$$

		$\begin{matrix} * \\ 2Iq_1\alpha \end{matrix}$						
	$\begin{matrix} * \\ -2Iq_1\alpha \end{matrix}$							

$$U_2 =$$

		$-M_b^*$			
$-2M_b^*$					
	$2M_b^*$				
					$H_o^*$
				$-H_o^*$	
			$H_o^*$		



$$\sum_1 =$$

			$2H_o^*$		
				$2H_o^*$	

$$\tilde{A}_0 = 0$$

$$\tilde{C}_0 = 0$$

and:

$$M_{aero} = \gamma \begin{bmatrix} \frac{M_{\theta_0}}{a/c} \\ \frac{M_{\theta_{ic}}}{a/c} \\ \frac{M_{\theta_{is}}}{a/c} \\ \frac{M_{\theta_0}}{a/c} \\ \frac{M_{\theta_{ic}}}{a/c} \\ \frac{M_{\theta_{is}}}{a/c} \\ -\frac{2C_{My}}{a/c} \\ \frac{2C_{Mx}}{a/c} \end{bmatrix}$$

$$F_{aero} = \gamma \begin{bmatrix} \frac{C_L}{a/c} \\ \frac{2C_D}{a/c} \\ -\frac{2C_z}{a/c} \\ \frac{C_L}{a/c} \\ \frac{2C_D}{a/c} \\ -\frac{2C_z}{a/c} \end{bmatrix}$$

## AERODYNAMICS

In this section, the aerodynamic forces and moments on the rotor blade are derived. We shall consider the general case of high or low inflow, and axial or nonaxial flow. The aerodynamic terms in the rotor equations of motion and the hub forces and moments are obtained for three cases: axial flow (hover or high inflow cruise); nonaxial flow with periodic coefficients (helicopter forward flight, or conversion mode flight), and a constant coefficient approximation for nonaxial flow.

The principle assumptions in the aerodynamic analysis are: reverse flow is neglected (good to an advance ratio of about 0.4 or 0.5, which is sufficient for the tilting proprotor aircraft); the wing wake (near field and far field) effect on the rotor, and other wing/rotor interferences are neglected; the unsteady rotor wake effects are neglected; the virtual mass aerodynamic forces and moments are neglected; the order  $c$  (rotor chord) terms in the aerodynamic lift expression are neglected; the order  $c^3$  terms in the aerodynamic moment expression are neglected; and only first order velocity terms are retained. The derivation and notation are an extension of that in reference 4.

### Section Aerodynamic Forces

A hub plane reference frame is used for the aerodynamic forces. All forces and velocities are resolved in the hub plane then, i.e. in the B system. The hub plane reference frame is fixed with respect to the shaft, hence it is tilted and displaced by the shaft motion. Figure 10 illustrates the forces and velocities of the blade section aerodynamics. The velocity of the air seen by the blade, the pitch angle, and the angle of attack are:

$\Theta$  = blade pitch, measured from the reference plane

$u_T, u_R, u_P$  = air velocity seen by the blade, resolved with respect to the reference plane;  $u_T$  is in the hub plane, positive in the blade drag direction;  $u_P$

is in the hub plane, positive radially outward along the blade; and  $u_p$  is normal to the hub plane, positive down through the rotor disk.

$U$  = resultant air velocity in the plane of the section.

$\phi$  = induced angle

$\alpha$  = section angle of attack

where

$$U^2 = u_r^2 + u_p^2$$

$$\phi = \tan^{-1} u_p / u_r$$

$$\alpha = \theta - \phi$$

The aerodynamic forces and moment on the section, at the EA, are:

$L, D$  = aerodynamic lift and drag forces on the section, normal and parallel to the resultant velocity  $U$

$F_z, F_x$  = section  $L$  and  $D$  (total aerodynamic force on the section) resolved with respect to the hub plane, normal to and in the plane of the rotor

$F_r$  = radial drag force on the blade, in the plane of the disk, positive outward (the same direction as positive  $u_r$ ); the radial forces due to the tilt of  $F_z$  and  $F_x$  have been considered separately.

$M_a$  = section aerodynamic moment about the EA, positive nose up.

#### Aerodynamic forces -- wind axes

The section lift and drag are

$$L = \frac{1}{2} \rho U^2 c c_L$$

$$D = \frac{1}{2} \rho U^2 c c_D$$

where

$U$  = resultant velocity at the section

$\rho$  = air density

$c$  = chord of blade

The air density is dropped at this point, in the process of making the quantities dimensionless with  $\rho, \Sigma$ , and  $R$ . The section lift

and drag coefficients,  $c_l = c_l(\alpha, M)$  and  $c_d = c_d(\alpha, M)$  are functions of the section angle of attack and Mach number:

$$\alpha = \theta - \phi = \theta - \tan^{-1} u_r/u_t$$

$$M = M_{TIP} U$$

where  $M_{TIP}$  is the tip Mach number, the rotor tip speed  $\Omega R$  divided by the speed of sound. The dependence of  $c_l$  and  $c_d$  on other quantities, such as the local yaw angle or unsteady angle of attack changes, is neglected. The radial force, due to the radial drag, is

$$F_r = \frac{u_r}{U} D = \frac{1}{2} U u_r c_d$$

The radial drag force is derived assuming that the viscous drag force on the section has the same sweep angle as the local section velocity. The moment about the EA is

$$\begin{aligned} M_a &= -x_A L + M_{ac} + M_{us} \\ &= -x_A \frac{1}{2} U^2 c_d + \frac{1}{2} U^2 c^2 c_{m_{ac}} + M_{us} \end{aligned}$$

where  $x_A$  = distance aerodynamic center (AC) behind EA  
 $c_{m_{ac}}$  = section moment about the AC, positive nose up.  
 $M_{us}$  = unsteady aerodynamic moment.

For the section aerodynamic moment it is necessary to include the unsteady aerodynamic terms, which from thin airfoil theory are

$$\begin{aligned} \frac{M_{us}}{a c} = & -\frac{\xi^2}{32} \left[ (VB) \left( 1 + 8 \frac{x_A}{c} + 16 \left( \frac{x_A}{c} \right)^2 \right) \right. \\ & \left. + (\dot{\omega} + u_r \omega') \left( 1 + 4 \frac{x_A}{c} \right) \right] \end{aligned}$$

where  $w$  = mean upwash along the blade chord, i.e. normal blade section  
 $= u_T \sin \theta - u_p \cos \theta$   
 $B = \partial w / \partial x$ , basically the pitch rate  $\dot{\theta}$   
 $V = u_T \cos \theta + u_p \sin \theta$

Hence in the aerodynamic model we have neglected the following effects: reverse flow; shed wake aerodynamic interference (e.g. lift deficiency function set to unity); terms in  $L$  order  $c$  and above; terms in  $M$  order  $c^2$  and above; virtual mass terms in the unsteady aerodynamic moment.

#### Aerodynamic forces -- hub plane axes

With respect to the hub plane then

$$F_z = L \cos \phi - D \sin \phi = \frac{L u_T - D u_P}{u}$$

$$F_x = L \sin \phi + D \cos \phi = \frac{L u_P + D u_T}{u}$$

Substituting for  $L$  and  $D$ , and dividing by  $ac$  (where  $a$  is the two-dimensional section lift curve slope, and  $c$  the section chord; which enter the Lock number  $\gamma$  also), we obtain:

$$\frac{F_z}{ac} = u \left( u_T \frac{C_L}{2a} - u_P \frac{C_D}{2a} \right)$$

$$\frac{F_x}{ac} = u \left( u_P \frac{C_L}{2a} + u_T \frac{C_D}{2a} \right)$$

$$\frac{F_r}{ac} = u u_R \frac{C_d}{2a}$$

$$\frac{M_a}{ac} = -\gamma u^2 \frac{C_L}{2a} + u^2 \frac{C_{Cm}}{2a} + \frac{M_{us}}{ac}$$

The net rotor aerodynamic forces are obtained by integration of the section forces over the span of the blade, and then summation over all  $N$  blades.

#### Perturbation forces

Each component of the velocity seen by the blade has a trim term, due to operation of the rotor in its trim equilibrium state; and a perturbation term due to the perturbed motion of the system. The latter is due to the system degrees of freedom, and is assumed to be small in obtaining the linear differential equations describing the dynamics. We shall write the blade

pitch and section velocities as trim plus perturbation terms:

$$\Theta \Rightarrow \Theta + \delta\Theta$$

$$u_T \Rightarrow u_T + \delta u_T$$

$$u_P \Rightarrow u_P + \delta u_P$$

$$u_R \Rightarrow u_R + \delta u_R$$

then there follows the perturbations of  $\alpha$ ,  $U$ , and  $M$ :

$$\delta\alpha = \delta\Theta - \frac{u_T \delta u_P - u_P \delta u_T}{U^2}$$

$$\delta U = \frac{u_T \delta u_T + u_P \delta u_P}{U}$$

$$\delta M = M_{TIP} \delta U$$

and of the aerodynamic coefficients

$$\delta C_L = \frac{\partial C_L}{\partial \alpha} \delta\alpha + \frac{\partial C_L}{\partial M} \delta M = C_{L_\alpha} \delta\alpha + C_{L_M} \delta M$$

(and similarly for  $c_m$  and  $c_d$ ). The perturbations of the section aerodynamic forces may then be obtained by carrying out the differential operation on the expressions above for  $F_z$ ,  $F_x$ ,  $F_r$ , and  $M_a$ , using the above results to express the perturbations in terms of  $\delta\Theta$ ,  $\delta u_T$ ,  $\delta u_P$ , and  $\delta u_R$ . The coefficients of the perturbation quantities are then evaluated at the trim state. The results are:

$$\begin{aligned} \delta \frac{F_z}{\rho c} = & \left( U u_T \frac{C_{L_\alpha}}{2a} - u_P \frac{C_{L_\alpha}}{2a} \right) \delta\Theta \\ & + \left[ -\frac{u_T}{U} \left( u_T \frac{C_{L_\alpha}}{2a} - u_P \frac{C_{L_\alpha}}{2a} \right) + \left( \frac{C_L}{2a} + M \frac{C_{L_M}}{2a} \right) \frac{u_T u_P}{U} \right. \\ & \quad \left. - \left( \frac{C_L}{2a} + M \frac{C_{L_M}}{2a} \right) \frac{u_P^2}{U} - \frac{C_L}{2a} U \right] \delta u_P \\ & + \left[ \frac{u_P}{U} \left( u_T \frac{C_{L_\alpha}}{2a} - u_P \frac{C_{L_\alpha}}{2a} \right) + \left( \frac{C_L}{2a} + M \frac{C_{L_M}}{2a} \right) \frac{u_T^2}{U} \right. \\ & \quad \left. + \frac{C_L}{2a} U - \left( \frac{C_L}{2a} + M \frac{C_{L_M}}{2a} \right) \frac{u_T u_P}{U} \right] \delta u_T \\ = & F_{z\Theta} \delta\Theta + F_{zP} \delta u_P + F_{zT} \delta u_T \end{aligned}$$

$$\begin{aligned}
\delta \frac{F_x}{\partial c} &= \left( u_{up} \frac{c_{\theta\theta}}{2a} + u_{ut} \frac{c_{dt}}{2a} \right) \delta \theta \\
&+ \left[ -\frac{u_t}{u} \left( u_p \frac{c_{\theta\theta}}{2a} + u_t \frac{c_{dt}}{2a} \right) + \left( \frac{c_\theta}{2a} + m \frac{c_{\theta m}}{2a} \right) \frac{u_p^2}{u} \right. \\
&\quad \left. + \frac{c_\theta}{2a} u + \left( \frac{c_d}{2a} + m \frac{c_{dm}}{2a} \right) \frac{u_t u_p}{u} \right] \delta u_p \\
&+ \left[ \frac{u_p}{u} \left( u_p \frac{c_{\theta\theta}}{2a} + u_t \frac{c_{dt}}{2a} \right) + \left( \frac{c_\theta}{2a} + m \frac{c_{\theta m}}{2a} \right) \frac{u_p u_t}{u} \right. \\
&\quad \left. + \left( \frac{c_d}{2a} + m \frac{c_{dm}}{2a} \right) \frac{u_t^2}{u} + \frac{c_d}{2a} u \right] \delta u_t \\
&= F_{x\theta} \delta \theta + F_{xp} \delta u_p + F_{xt} \delta u_t
\end{aligned}$$

$$\begin{aligned}
\delta \frac{F_r}{\partial c} &= \left( u_{ur} \frac{c_{\theta\theta}}{2a} \right) \delta \theta \\
&+ \left[ -\frac{u_t u_r}{u} \frac{c_{dt}}{2a} + \left( \frac{c_d}{2a} + m \frac{c_{dm}}{2a} \right) \frac{u_r u_p}{u} \right] \delta u_p \\
&+ \left[ \frac{u_p u_r}{u} \frac{c_{dt}}{2a} + \left( \frac{c_d}{2a} + m \frac{c_{dm}}{2a} \right) \frac{u_r u_t}{u} \right] \delta u_t \\
&+ \left[ u \frac{c_d}{2a} \right] \delta u_r \\
&= F_{r\theta} \delta \theta + F_{rp} \delta u_p + F_{rt} \delta u_t + F_{rr} \delta u_r
\end{aligned}$$



and:

$$\begin{aligned}
 \delta \frac{M_a}{a c} = & \left[ u^2 \left( -x_A \frac{C_{L\alpha}}{2a} + c \frac{C_{M\alpha}}{2a} \right) \right] \delta \theta \\
 & + \left[ u_P \left( -x_A \frac{C_{L\alpha}}{2a} + c \frac{C_{M\alpha}}{2a} \right) - x_A u_T \left( 2 \frac{C_L}{2a} + M \frac{C_{Lm}}{2a} \right) \right. \\
 & \quad \left. + c u_T \left( 2 \frac{C_m}{a c} + M \frac{C_{mm}}{2a} \right) \right] \delta u_T \\
 & + \left[ -u_T \left( -x_A \frac{C_{L\alpha}}{2a} + c \frac{C_{M\alpha}}{2a} \right) - x_A u_P \left( 2 \frac{C_L}{2a} + M \frac{C_{Lm}}{2a} \right) \right. \\
 & \quad \left. + c u_P \left( 2 \frac{C_m}{a c} + M \frac{C_{mm}}{a c} \right) \right] \delta u_P \\
 & + \frac{M u_S}{a c} \\
 = & M_{a\theta} \delta \theta + M_{aP} \delta u_P + M_{aT} \delta u_T + \frac{M u_S}{a c}
 \end{aligned}$$

### Velocity of the Blade

Now we obtain the velocity of the air seen by the blade section. There is the trim velocity, composed of the forward speed, rotor rotation, and rotor induced velocity; and the perturbation velocities, due to the rotor degrees of freedom and the shaft motion, and due to the aerodynamic gust velocity.

The rotor is rotating at constant speed  $\Omega$ . The steady velocity of the rotor with respect to the air, is described by (figure 11):

$V$  = trim velocity of the rotor in inertial axes, in the rotor x-z plane.

$\alpha_{hp}$  = angle of attack (undisturbed) of the rotor hub plane with respect to  $V$ , positive for disk tilt forward (for  $V$  down through the disk); this is the shaft angle.

There are then the following cases:  $\alpha_{HP} = 90^\circ$  for cruise (high inflow axial flight);  $\alpha_{HP}$  small for helicopter forward flight;  $\alpha_{HP}$  large but less than  $90^\circ$  for conversion mode; and  $V = 0$  is the hover case. The rotor induced velocity is  $v$ , due to the thrust  $T$  (figure 11);  $v$  is assumed to be normal to the hub plane, and uniform over the disk. Now the rotor advance ratio  $\mu$  and inflow ratio  $\lambda$  are defined:

$$\mu = \frac{V \cos \alpha_{HP}}{\Omega R}$$

$$\lambda = \frac{V \sin \alpha_{HP} + v}{\Omega R}$$

The cases are then: for hover  $\mu = 0$  and  $\lambda$  small; for helicopter forward flight  $\mu \neq 0$  and  $\lambda$  small; for conversion mode flight  $\mu \neq 0$  and  $\lambda$  order 1; and for cruise flight  $\mu = 0$  and  $\lambda$  order 1.

For the rotor induced velocity we use the Glauert result:

$$\lambda = \mu \tan \alpha_{HP} + \frac{C_T}{2\sqrt{\mu^2 + \lambda^2}}$$

For high speed ( $V^2 \gg \frac{1}{2} C_T (\Omega R)^2$  or about  $V/\Omega R > 0.15$ ) in inflow ratio is approximately

$$\lambda = \frac{V}{\Omega R} \sin \alpha_{HP} + \frac{C_T}{2 V/\Omega R}$$

The induced velocity is thus quite small,  $v/V \ll 1$ , for typical proprotor cruise and conversion mode operation. The induced velocity is not generally an important factor in proprotor aerodynamics at high inflow; hence the assumption of uniform induced inflow is acceptable for an investigation of the proprotor aeroelastic behavior. (See reference 4.) The mutual aerodynamic interference of the rotors is neglected.

The trim velocity  $V$  is steady, at an angle  $\alpha_{HP}$  to the rotor hub plane. The uniform induced velocity  $v$  is normal to the hub plane. The advance ratio and inflow ratio,  $\mu$  and  $\lambda$ , are the nondimensional

components parallel and normal to the hub plane. In body axes,  $V$  would be fixed in the reference frame, and would tilt with it. Here an inertial frame (the S system) is used however, so it follows that tilt of the rotor by the shaft motion gives a small change in the direction of  $V$  as seen in the reference frame.

The shaft motion consists of small linear and angular velocity, with components defined in the nonrotating frame:

$$\begin{aligned}\Delta \vec{V}_0 &= \dot{x}_0 \vec{e}_s + \dot{y}_0 \vec{e}_s + \dot{z}_0 \vec{e}_s \\ \vec{\omega}_0 &= \dot{\alpha}_x \vec{e}_s + \dot{\alpha}_y \vec{e}_s + \dot{\alpha}_z \vec{e}_s\end{aligned}$$

The aerodynamic gust velocity has components  $u_G$ ,  $v_G$ , and  $w_G$  (longitudinal, lateral, and vertical) defined with respect to the body or earth axes (figure 11); these components are the velocity seen by the aircraft, and are assumed to be small compared to  $\Omega R$ . The gust components are defined with respect to  $V$ , i.e.  $\alpha_{WP}$  from the disk plane, so that with  $V$  usually horizontal (level flight)  $w_G$  and  $u_G$  are always the vertical and longitudinal components with respect to the flight path. The gust components are normalized by dividing by  $\Omega R$ , not by  $V$  as is often the convention for airplane analyses. The aerodynamic gust is assumed to be uniform throughout space.

#### Trim terms

The result for the trim velocity terms is:

$$\begin{aligned}u_T &= r + \mu \sin \psi - \mu \cos \psi (\delta \alpha_1 - \epsilon q_1 \vec{e}_0 \cdot \vec{\eta}_1') \\ &\quad + \epsilon q_1 \vec{e}_0 \cdot \vec{\eta}_1' \\ u_P &= \lambda + \epsilon q_1 \vec{e}_0 \cdot \vec{\eta}_1' + r(\beta_0 \\ &\quad + \mu \cos \psi (\delta \alpha_1 - \delta \alpha_2 + \epsilon q_1 \vec{e}_0 \cdot \vec{\eta}_1' + \beta_0))\end{aligned}$$

$$\begin{aligned}
 U_R &= \mu \cos \psi + (x_{FA} + r_{FA} \delta FA_1) + \sum q_i \vec{k}_B \cdot (\vec{\eta}_i - r \vec{\eta}_i^v) \\
 &\rightarrow (\delta FA_1 - \delta FA_2 + \sum q_i \vec{k}_B \cdot \vec{\eta}_i^v + \beta_G) \\
 &\quad + \mu \sin \psi (\delta FA_3 - \sum q_i \vec{k}_B \cdot \vec{\eta}_i^v)
 \end{aligned}$$

and

$$\Theta = \Theta_{\omega 11} + \Theta_{\pi \omega} + \Theta_{cyc} - K_{\beta G} \beta_G - \sum K_{\beta q_i} q_i$$

where  $\Theta_{cyc}$  is the input cyclic pitch required to trim the rotor. For the trim velocity, the blade bending and gimbal motion is periodic. For axial flight,  $\mu = 0$ , the trim velocities are constant; for nonaxial flow,  $\mu > 0$ , these velocities are periodic in  $\psi_m$ , due to the rotation of the blade with respect to the rotor forward velocity.

#### Perturbation terms

The result for the perturbations of the velocity components and the blade pitch, due to the rotor and shaft motion and the aerodynamic gust, is then:

$$\begin{aligned}
 \delta u_T &= (\lambda \alpha_x + \dot{y}_u + v_0) \cos \psi_m \\
 &\quad + (\lambda \alpha_y - \dot{x}_u + v_0 \cos \alpha_{HP} + \omega_0 \sin \alpha_{HP}) \sin \psi_m \\
 &\quad + \mu \cos \psi_m (\alpha_z + \psi_s) \\
 &\quad + r (\dot{\alpha}_z + \dot{\psi}_s) \\
 &\quad + \sum \dot{q}_i (\vec{k}_B \cdot \vec{\eta}_i^v) \\
 &\quad + \mu \cos \psi_m \sum q_i (\vec{k}_B \cdot \vec{\eta}_i^v)
 \end{aligned}$$

$$\begin{aligned}
\delta u_p = & (\dot{z}_h - \mu \alpha_y + u_G \sin \alpha_{hp} - \omega_G \cos \alpha_{hp}) \\
& + \mu \cos \psi_m (\beta_G) \\
& + r (\dot{\beta}_G + \dot{\alpha}_x \sin \psi_m - \dot{\alpha}_y \cos \psi_m) \\
& + \sum \dot{q}_i (\vec{r}_B \cdot \vec{\eta}_i) \\
& + \mu \cos \psi_m \sum q_i (\vec{r}_B \cdot \vec{\eta}_i^{\prime\prime})
\end{aligned}$$

$$\begin{aligned}
\delta u_R = & - (\lambda \alpha_x + \dot{y}_h + v_G) \sin \psi_m \\
& + (\lambda \alpha_y - \dot{x}_h + u_G \cos \alpha_{hp} + \omega_G \sin \alpha_{hp}) \cos \psi_m \\
& - \lambda \beta_G - \mu \sin \psi_m (\alpha_z + \psi_s) \\
& + \sum q_i [ \vec{r}_B \cdot (\vec{\eta}_i - r \vec{\eta}_i^{\prime} - \mu \sin \psi_m \vec{\eta}_i^{\prime\prime}) - \lambda \vec{r}_B \cdot \vec{\eta}_i^{\prime\prime} ]
\end{aligned}$$

$$\delta \theta = \tilde{\theta} = \sum p_i \xi_i$$

and for  $M_{hs}$

$$v_B = (u_T \cos \theta + u_R \sin \theta) (\dot{\theta} + \beta_G + \sum q_i \vec{r}_B \cdot \vec{\eta}_i^{\prime\prime})$$

$$\begin{aligned}
\dot{\omega} + u_R \omega^{\prime\prime} = & \sum \dot{p}_i \xi_i (u_T \cos \theta + u_R \sin \theta) \\
& + u_R \sum p_i ( \xi_i^{\prime} (u_T \cos \theta + u_R \sin \theta) + 2 \xi_i \cos \theta ) \\
& - \dot{\beta}_G (2 u_R \cos \theta) \\
& + \beta_G (\mu \sin \psi_m \cos \theta) \\
& + (\dot{\alpha}_z + \dot{\psi}_s) (2 u_R \sin \theta) \\
& - (\alpha_z + \psi_s) (\mu \sin \psi_m \sin \theta) \\
& - 2 u_R \sum \dot{q}_i \vec{r} \cdot \vec{\eta}_i^{\prime} \\
& + \sum q_i (\mu \sin \psi_m \vec{r} \cdot \vec{\eta}_i^{\prime\prime} - u_R^2 \vec{r} \cdot \vec{\eta}_i^{\prime\prime\prime} )
\end{aligned}$$

### Rotor Aerodynamic Forces -- Rotating Blade

With now the expansions for the section forces and moment in terms of the velocity perturbations, and the velocity in terms of the motion of the rotor and shaft, we may obtain the perturbations of the aerodynamic forces on the blade. These are the blade forces expanded as linear combinations of the degrees of freedom. Giving names to the aerodynamic coefficients at this point, the results for the required aerodynamic forces on the rotating blade are as follows.

Bending:

$$\begin{aligned} \int_0^1 \vec{\eta}_k \cdot \left( \frac{F_z}{a_c} \vec{e}_z - \frac{F_x}{a_c} \vec{e}_x \right) dr = \\ M_{qk0} + M_{qk\mu} [ (\lambda \alpha_x + \dot{y}_h + v_0) \cos \psi_m \\ + (\lambda \alpha_y - \dot{z}_h + u_0 \cos \alpha_{hp} + w_0 \sin \alpha_{hp}) \sin \psi_m ] \\ + M_{qk\dot{z}} (\dot{\alpha}_z + \dot{\psi}_s) \\ + M_{qk\zeta} (\alpha_z + \psi_s) \\ + M_{qk\lambda} (\dot{z}_h - \mu \alpha_y + u_0 \sin \alpha_{hp} - w_0 \cos \alpha_{hp}) \\ + M_{qk\dot{\beta}} (\dot{\beta}_0 + \dot{\alpha}_x \sin \psi_m - \dot{\alpha}_y \cos \psi_m) \\ + M_{qk\beta} \beta_0 \\ + \sum M_{qk\dot{q}_i} \dot{q}_i \\ + \sum M_{qkq_i} q_i \\ + \sum M_{qkp_i} p_i \end{aligned}$$

radial force:

$$\begin{aligned}
 \int_0^1 \frac{F_r}{a^2} - \frac{F_z}{a^2} (\beta_0 + \delta_1 - \delta_2 + \vec{k}_B \cdot (\vec{x}_0 \vec{e} + \vec{z}_0 \vec{k})^v) dr \\
 = R_\mu [ -(\lambda \alpha_x + \dot{y}_\mu + v_0) \sin \psi_\mu \\
 + (\lambda \alpha_y - \dot{x}_\mu + u_0 \cos \alpha_{\mu\theta} + \omega_0 \sin \alpha_{\mu\theta}) \cos \psi_\mu ] \\
 + R_r [ (\lambda \alpha_x + \dot{y}_\mu + v_0) \cos \psi_\mu \\
 + (\lambda \alpha_y - \dot{x}_\mu + u_0 \cos \alpha_{\mu\theta} + \omega_0 \sin \alpha_{\mu\theta}) \sin \psi_\mu ] \\
 + R_{\dot{z}} (\dot{\alpha}_z + \dot{\psi}_s) + R_{\dot{y}} (\alpha_z + \psi_s) \\
 + R_{\dot{x}} (\dot{z}_\mu - \mu \alpha_y + u_0 \sin \alpha_{\mu\theta} - \omega_0 \cos \alpha_{\mu\theta}) \\
 + R_{\dot{\beta}} (\dot{\beta}_0 + \dot{\alpha}_x \sin \psi_\mu - \dot{\alpha}_y \cos \psi_\mu) + R_{\beta} \beta_0 \\
 + \sum R_{\dot{q}_i} \dot{q}_i + \sum R_{q_i} q_i \\
 + \sum R_{\dot{p}_i} \dot{p}_i
 \end{aligned}$$

Torsion/pitch:

$$\begin{aligned}
 \int_{r_{\mu\lambda}}^1 \xi_k M_{\mu\lambda} dr - \int_{r_{\mu\lambda}}^1 \left( \frac{F_x}{a^2} \vec{e}_B + \frac{F_z}{a^2} \vec{k}_B \right) \cdot \vec{\chi}_{A\mu} dr \\
 = M_{\mu\lambda} [ (\lambda \alpha_x + \dot{y}_\mu + v_0) \cos \psi_\mu \\
 + (\lambda \alpha_y - \dot{x}_\mu + u_0 \cos \alpha_{\mu\theta} + \omega_0 \sin \alpha_{\mu\theta}) \sin \psi_\mu ] \\
 + M_{\mu\lambda} (\dot{\alpha}_z + \dot{\psi}_s) + M_{\mu\lambda} (\alpha_z + \psi_s) \\
 + M_{\mu\lambda} [\dot{z}_\mu - \mu \alpha_y + u_0 \sin \alpha_{\mu\theta} - \omega_0 \cos \alpha_{\mu\theta}] \\
 + M_{\mu\lambda} (\dot{\beta}_0 + \dot{\alpha}_x \sin \psi_\mu - \dot{\alpha}_y \cos \psi_\mu) + M_{\mu\lambda} \beta_0 \\
 + \sum M_{\mu\lambda} \dot{q}_i \dot{q}_i + \sum M_{\mu\lambda} q_i q_i \\
 + \sum M_{\mu\lambda} \dot{p}_i \dot{p}_i + \sum M_{\mu\lambda} p_i p_i
 \end{aligned}$$

Hub forces and moments: similar to the bending case, but with notation

	<u>integrand</u>	<u>notation</u>
flap moment	$rF_z$	M
torque	$rF_x$	Q
blade drag force	$F_x$	H
thrust	$F_z$	T

### Aerodynamic coefficients

Applying the results for the expansion of the aerodynamic forces, and the expansion of the velocities, the aerodynamic coefficients may be evaluated. These coefficients of the degrees of freedom in the aerodynamic forces are constant for axial flow, the  $\mu = 0$  case. For the general nonaxial flow case,  $\mu > 0$ , the coefficients are however periodic functions of  $\psi_m$ . The results follow.

Bending:

$$M_{qk0} = \int_0^1 \tilde{\eta}_k \cdot \left( \frac{F_z}{a_c} \tau_0 - \frac{F_x}{a_c} \tau_0 \right) dr$$

$$M_{qk\mu} = \int_0^1 \tilde{\eta}_k \cdot (F_{z\tau} \tau_0 - F_{x\tau} \tau_0) dr$$

$$M_{qk\beta} = \int_0^1 \tilde{\eta}_k \cdot (F_{z\tau} \tau_0 - F_{x\tau} \tau_0) r dr$$

$$M_{qk\beta} = \mu \cos \psi_m M_{qk\mu}$$

$$M_{qk\lambda} = \int_0^1 \tilde{\eta}_k \cdot (F_{z\tau} \tau_0 - F_{x\tau} \tau_0) dr$$

$$M_{qk\beta} = \int_0^1 \tilde{\eta}_k \cdot (F_{z\tau} \tau_0 - F_{x\tau} \tau_0) r dr$$

$$M_{qk\beta} = \mu \cos \psi_m M_{qk\lambda}$$

$$M_{qk\dot{q}_i} = \int_0^1 \tilde{\eta}_k \cdot (F_{z\tau} \tau_0 - F_{x\tau} \tau_0) \tau_0 \cdot \dot{\eta}_i dr \\ + \int_0^1 \tilde{\eta}_k \cdot (F_{z\tau} \tau_0 - F_{x\tau} \tau_0) \tau_0 \cdot \dot{\eta}_i dr$$

$$M_{qk\dot{q}_i} = \mu \cos \psi_m \left[ \int_0^1 \tilde{\eta}_k \cdot (F_{z\tau} \tau_0 - F_{x\tau} \tau_0) \tau_0 \cdot \dot{\eta}_i dr \right. \\ \left. + \int_0^1 \tilde{\eta}_k \cdot (F_{z\tau} \tau_0 - F_{x\tau} \tau_0) \tau_0 \cdot \dot{\eta}_i dr \right]$$

$$M_{qk\dot{p}_i} = \int_0^1 \tilde{\eta}_k \cdot (F_{z0} \tau_0 - F_{x0} \tau_0) \dot{\eta}_i dr$$



Flap moment:

$$M_\mu = \int_0^1 F_{zT} r dr$$

$$M_{\dot{\beta}} = \int_0^1 F_{zT} r^2 dr$$

$$M_{\dot{\beta}} = \mu \cos \psi_m M_\mu$$

$$M_\lambda = \int_0^1 F_{zP} r dr$$

$$M_{\dot{\beta}} = \int_0^1 F_{zP} r^2 dr$$

$$M_{\dot{\beta}} = \mu \cos \psi_m M_\lambda$$

$$M_{\dot{\beta}} = \int_0^1 (F_{zT} \vec{e}_B \cdot \vec{\eta}_i + F_{zP} \vec{e}_B \cdot \vec{\eta}_i) r dr$$

$$M_{\dot{\beta}} = \mu \cos \psi_m \int_0^1 (F_{zT} \vec{e}_B \cdot \vec{\eta}_i + F_{zP} \vec{e}_B \cdot \vec{\eta}_i) r dr$$

$$M_P = \int_0^1 F_{z\theta} \xi_i r dr$$

Other hub forces and moments: similar to flap moment, with

	<u>coefficient</u>	<u>integrand</u>
flap moment	$\mu$	$r F_z$
torque	$\mu$	$r F_x$
blade drag force	$H$	$F_x$
thrust	$T$	$F_z$

Radial force:

$$R_\mu = \int_0^1 F_{rT} dr$$

$$R_r = \int_0^1 [F_{rT} - F_{zT} (\delta_1 - \delta_2 + \vec{e}_B \cdot (\vec{x}_0 \vec{t} + \vec{z}_0 \vec{k})^v)] dr$$

$$R_{\dot{\beta}} = \int_0^1 [F_{rT} - F_{zT} (\delta_1 - \delta_2 + \vec{e}_B \cdot (\vec{x}_0 \vec{t} + \vec{z}_0 \vec{k})^v)] r dr$$

$$R_{\dot{\beta}} = \mu \cos \psi_m R_r - \mu \sin \psi_m R_\mu$$

$$R_\lambda = \int_0^1 [F_{rP} - F_{zP} (\delta_1 - \delta_2 + \vec{e}_B \cdot (\vec{x}_0 \vec{t} + \vec{z}_0 \vec{k})^v)] dr$$

$$R_{\dot{\beta}} = \int_0^1 [F_{rP} - F_{zP} (\delta_1 - \delta_2 + \vec{e}_B \cdot (\vec{x}_0 \vec{t} + \vec{z}_0 \vec{k})^v)] r dr$$

$$R_\beta = \mu \cos \psi_m R_\lambda - \lambda R_\mu - \int_0^1 \frac{F_z}{ac} dr$$

$$R_{q_i} = \int_0^1 [F_{r_T} - F_{z_T}(\delta_1 - \delta_2 + \vec{k}_B \cdot (x_0 \vec{r} + z_0 \vec{r})^v)] \vec{E}_B \cdot \vec{q}_i dr \\ + \int_0^1 [F_{r_P} - F_{z_P}(\delta_1 - \delta_2 + \vec{k}_B \cdot (x_0 \vec{r} + z_0 \vec{r})^v)] \vec{z}_B \cdot \vec{q}_i dr$$

$$R_{q_i} = \mu \cos \psi_m \left\{ \int_0^1 [F_{r_T} - F_{z_T}(\delta_1 - \delta_2 + \vec{k}_B \cdot (x_0 \vec{r} + z_0 \vec{r})^v)] \vec{E}_B \cdot \vec{q}_i dr \right. \\ \left. + \int_0^1 [F_{r_P} - F_{z_P}(\delta_1 - \delta_2 + \vec{k}_B \cdot (x_0 \vec{r} + z_0 \vec{r})^v)] \vec{z}_B \cdot \vec{q}_i dr \right\} \\ + \int_0^1 F_{r_R} [\vec{E}_B \cdot (\vec{q}_i - r \vec{q}_i' - \mu \sin \psi_m \vec{q}_i') - \lambda \vec{z}_B \cdot \vec{q}_i'] dr \\ - \int_0^1 \frac{F_z}{ac} \vec{z}_B \cdot \vec{q}_i' dr$$

$$R_{p_i} = \int_0^1 [F_{r_\theta} - F_{z_\theta}(\delta_1 - \delta_2 + \vec{k}_B \cdot (x_0 \vec{r} + z_0 \vec{r})^v)] \vec{q}_i dr$$

Torsion/pitch:

$$M_{PK\mu} = \int_{r_{PA}}^1 [\beta_k M_{aT} - (F_{x_T} \vec{z}_B + F_{z_T} \vec{E}_B) \cdot \vec{X}_{AK}] dr$$

$$M_{PK\dot{\xi}} = \int_{r_{PA}}^1 [\beta_k M_{aT} - (F_{x_T} \vec{z}_B + F_{z_T} \vec{E}_B) \cdot \vec{X}_{AK}] r dr \\ - \int_{r_{PA}}^1 \beta_k \frac{\xi^2}{32} (1 + 4 \frac{\chi_A}{c}) 2 \mu R \sin \theta dr$$

$$M_{PK\zeta} = \mu \cos \psi_m M_{PK\mu} \\ + \int_{r_{PA}}^1 \beta_k \frac{\xi^2}{32} (1 + 4 \frac{\chi_A}{c}) \mu \sin \psi_m \sin \theta dr$$

$$M_{PK\lambda} = \int_{r_{PA}}^1 [\beta_k M_{aP} - (F_{x_P} \vec{z}_B + F_{z_P} \vec{E}_B) \cdot \vec{X}_{AK}] dr$$

$$M_{PK\dot{\beta}} = \int_{r_{PA}}^1 [\beta_k M_{aP} - (F_{x_P} \vec{z}_B + F_{z_P} \vec{E}_B) \cdot \vec{X}_{AK}] r dr \\ + \int_{r_{PA}}^1 \beta_k \frac{\xi^2}{32} (1 + 4 \frac{\chi_A}{c}) 2 \mu R \cos \theta dr$$

$$M_{PK\beta} = \mu \cos \psi_m M_{PK\lambda} \\ - \int_{r_{PA}}^1 \beta_k \frac{\xi^2}{32} (\mu R \cos \theta + \mu R \sin \theta) (1 + 8 \frac{\chi_A}{c} + 16 (\frac{\chi_A}{c})^2) dr \\ - \int_{r_{PA}}^1 \beta_k \frac{\xi^2}{32} (1 + 4 \frac{\chi_A}{c}) \mu \sin \psi_m \cos \theta dr$$

$$M_{PKq_i} = \int_{r_{PA}}^1 [\xi_k M_{aT} - (F_{xT} \vec{e}_B + F_{zT} \vec{e}_B) \cdot \vec{X}_{AK}] \vec{e}_B \cdot \vec{\eta}_i \, dr \\ + \int_{r_{PA}}^1 [\xi_k M_{aP} - (F_{xP} \vec{e}_B + F_{zP} \vec{e}_B) \cdot \vec{X}_{AK}] \vec{e}_B \cdot \vec{\eta}_i \, dr \\ + \int_{r_{PA}}^1 \xi_k \frac{c^2}{32} (1 + 4 \frac{x_A}{c}) 2u_R \vec{e}_B \cdot \vec{\eta}_i \, dr$$

$$M_{PKq_i} = \mu \cos \psi_m \left\{ \int_{r_{PA}}^1 [\xi_k M_{aT} - (F_{xT} \vec{e}_B + F_{zT} \vec{e}_B) \cdot \vec{X}_{AK}] \vec{e}_B \cdot \vec{\eta}_i \, dr \right. \\ \left. + \int_{r_{PA}}^1 [\xi_k M_{aP} - (F_{xP} \vec{e}_B + F_{zP} \vec{e}_B) \cdot \vec{X}_{AK}] \vec{e}_B \cdot \vec{\eta}_i \, dr \right\} \\ - \int_{r_{PA}}^1 (\frac{F_x}{a_c} \vec{e}_B + \frac{F_z}{a_c} \vec{e}_B) \cdot \vec{X}_{AKq_i} \, dr \\ - \int_{r_{PA}}^1 \xi_k \frac{c^2}{32} (u_T \cos \theta + u_P \sin \theta) (1 + 8 \frac{x_A}{c} + 16 (\frac{x_A}{c})^2) \vec{e}_B \cdot \vec{\eta}_i \, dr \\ - \int_{r_{PA}}^1 \xi_k \frac{c^2}{32} (1 + 4 \frac{x_A}{c}) (\mu \sin \psi \vec{e}_B \cdot \vec{\eta}_i - (\mu \cos \psi)^2 \vec{e}_B \cdot \vec{\eta}_i) \, dr$$

$$M_{PKP_i} = \int_{r_{PA}}^1 [\xi_k M_{a\theta} - (F_{x\theta} \vec{e}_B + F_{z\theta} \vec{e}_B) \cdot \vec{X}_{AK}] \xi_i \, dr \\ - \int_{r_{PA}}^1 \xi_k \frac{c^2}{32} u_R [\xi_i (u_T \cos \theta + u_P \sin \theta) + 2 \xi_i \cos \theta] (1 + 4 \frac{x_A}{c}) \, dr$$

$$M_{PKP_i} = - \int_{r_{PA}}^1 \xi_k \xi_i \frac{c^2}{16} (u_T \cos \theta + u_P \sin \theta) (1 + 6 \frac{x_A}{c} + 8 (\frac{x_A}{c})^2) \, dr$$

where

$$\vec{X}_{AK} = - \int_{r_{PA}}^r \xi_k (\vec{e}_B \cdot \vec{x}_0 \vec{e}_B)^{vv} (r-s) \, ds \\ \vec{X}_{A0} = - (\vec{e}_B \cdot \vec{x}_0 \vec{e}_B) + (\delta_2 \vec{e}_B + \delta_3 \vec{e}_B) (r - r_{PA}) \\ + (\vec{e}_B \cdot \vec{x}_0 \vec{e}_B)|_{r_{PA}} + (\vec{e}_B \cdot \vec{x}_0 \vec{e}_B)'|_{r_{PA}} (r - r_{PA})$$

and

$$\vec{X}_{AKq_i} = - \int_{r_{PA}}^r \xi_k \vec{\eta}_i^{vv} (r-s) \, ds \\ \vec{X}_{A0q_i} = - \vec{\eta}_i + \vec{\eta}_i(r_{PA}) + \vec{\eta}_i'(r_{PA}) (r - r_{PA})$$

### Rotor Aerodynamic Forces -- Nonrotating Frame

The aerodynamic forcing functions for the rotor equations of motion in the nonrotating frame, and the hub forces and moments are now required. These are obtained by summing the blade rotating forces (given above) over all N blades. The Fourier coordinate transform of the rotor degrees of freedom is introduced as required.

#### Axial Flow

First consider the case of axial flow,  $\mu = 0$ ; for either high inflow ratio  $\lambda$  (order 1, i.e. proprotor cruise flight), or low inflow (small  $\lambda$ , i.e. hover in helicopter mode). In this case the aerodynamic coefficients in the blade forces are constant, independent of  $M_{\infty}$ . The coefficients are also independent of m (the blade index) then, so the summation over N blades operates only on the blade degrees of freedom and shaft motion variables. The result for the required aerodynamic forces, in matrix form, is

$$\begin{aligned} -M_{aero} &= A_1 \dot{x}_R + A_0 x_R + \tilde{A}_1 \dot{\alpha} + \tilde{A}_0 \alpha - B_G g \\ F_{aero} &= C_1 \dot{x}_R + C_0 x_R + \tilde{C}_1 \dot{\alpha} + \tilde{C}_0 \alpha + D_G g \end{aligned}$$

where the rotor degrees of freedom ( $\vec{x}_R$ ), shaft motion ( $\vec{\alpha}$ ), and aerodynamic gust input ( $\vec{g}$ ) are:

$$\vec{x}_R = \begin{bmatrix} \beta_r \\ \beta_{1c} \\ \beta_{1s} \\ \theta_0 \\ \theta_{1c} \\ \theta_{1s} \\ \beta_{ac} \\ \beta_{as} \\ \psi_s \end{bmatrix} \quad \vec{\alpha} = \begin{bmatrix} x_h \\ y_h \\ z_h \\ \phi_y \\ \phi_z \end{bmatrix} \quad \vec{g} = \begin{bmatrix} u_G \\ v_G \\ w_G \end{bmatrix}$$

These coefficients simply add to the inertial coefficients already derived, to complete the equations of motion. The matrices of the aerodynamic coefficients follow.

$$A_1 =$$

$-\delta M_{qk\dot{q}_i}$								$-\delta M_{qk\dot{z}}$
	$-\delta M_{qk\dot{q}_i}$					$-\delta M_{qk\dot{p}}$		
		$-\delta M_{qk\dot{q}_i}$					$-\delta M_{qk\dot{p}}$	
$-\delta M_{pk\dot{q}_i}$			$-\delta M_{pk\dot{p}}$					$-\delta M_{pk\dot{z}}$
	$-\delta M_{pk\dot{q}_i}$			$-\delta M_{pk\dot{p}}$		$-\delta M_{pk\dot{p}}$		
		$-\delta M_{pk\dot{q}_i}$			$-\delta M_{pk\dot{p}}$		$-\delta M_{pk\dot{p}}$	
	$-\delta M_{q_i}$					$-\delta M_{\dot{p}}$		
		$-\delta M_{q_i}$					$-\delta M_{\dot{p}}$	

$$A_0 =$$

$-\delta M_{\epsilon, kq}$			$-\delta M_{qkp}$					$-\delta M_{q\mu 5}$
	$-\delta M_{qkq}$	$-\delta M_{q\mu q}$		$-\delta M_{qkp}$		$-\delta M_{q\mu p}$	$-\delta M_{q\mu \dot{p}}$	
	$\delta M_{q\mu q}$	$-\delta M_{q\mu q}$			$-\delta M_{qkp}$	$\delta M_{q\mu \dot{p}}$	$-\delta M_{q\mu p}$	
$-\delta M_{p\mu q}$			$-\delta M_{p\mu p}$					$-\delta M_{p\mu 5}$
	$-\delta M_{p\mu q}$	$-\delta M_{p\mu q}$		$-\delta M_{p\mu p}$	$-\delta M_{p\mu \dot{p}}$	$-\delta M_{p\mu p}$	$-\delta M_{p\mu \dot{p}}$	
	$\delta M_{p\mu q}$	$-\delta M_{p\mu q}$		$\delta M_{p\mu \dot{p}}$	$-\delta M_{p\mu p}$	$\delta M_{p\mu \dot{p}}$	$-\delta M_{p\mu p}$	
	$-\delta M_q$	$-\delta M_{\dot{q}}$		$-\delta M_p$		$-\delta M_{\dot{p}}$	$-\delta M_{\dot{p}}$	
	$\delta M_{\dot{q}}$	$-\delta M_q$			$-\delta M_p$	$\delta M_{\dot{p}}$	$-\delta M_p$	

$$\tilde{A}_i =$$

		$-\delta M_{qk\lambda}$			$-\delta M_{qk\xi}$
	$-\delta M_{qkr}$			$\delta M_{qk\beta}$	
$\delta M_{qk\mu}$			$-\delta M_{qk\dot{\rho}}$		
		$-\delta M_{rk\lambda}$			$-\delta M_{rk\xi}$
	$-\delta M_{rk\mu}$			$\delta M_{rk\beta}$	
$\delta M_{rk\rho}$			$-\delta M_{rk\dot{\rho}}$		
	$-\delta M_r$			$\delta M_{\dot{\rho}}$	
$\delta M_r$			$-\delta M_{\dot{\rho}}$		

$$\tilde{A}_0 =$$

				$\delta_\mu M_{q\kappa\lambda}$	$-\delta M_{q\kappa\zeta}$
			$-\delta\lambda M_{q\mu\mu}$		
				$-\delta\lambda M_{q\mu\mu}$	
				$\delta_\mu M_{p\kappa\lambda}$	$-\delta M_{p\kappa\zeta}$
			$-\delta\lambda M_{p\mu\mu}$		
				$-\delta\lambda M_{p\mu\mu}$	
			$-\delta\lambda M_\mu$		
				$-\delta\lambda M_\mu$	



$$B_G =$$

$\delta \sin \alpha M_{qk\lambda}$		$-\delta \cos \alpha M_{qk\lambda}$
	$\delta M_{qkp}$	
$-\delta \cos \alpha M_{qkp}$		$\delta \sin \alpha M_{qkp}$
$\delta \sin \alpha M_{pk\lambda}$		$-\delta \cos \alpha M_{pk\lambda}$
	$\delta M_{pkp}$	
$\delta \cos \alpha M_{pkp}$		$\delta \sin \alpha M_{pkp}$
	$\delta M_p$	
$\delta \cos \alpha M_p$		$\delta \sin \alpha M_p$

$$C_1 =$$

$\delta T_{\dot{q}_i}$								$\delta T_{\dot{q}_j}$
	$\delta R_{\dot{q}_i}$	$\delta H_{\dot{q}_i}$				$\delta R_{\dot{p}_j}$	$\delta H_{\dot{p}_j}$	
	$\delta H_{\dot{q}_i}$	$-\delta R_{\dot{q}_i}$				$\delta H_{\dot{p}_j}$	$-\delta R_{\dot{p}_j}$	
$\delta Q_{\dot{q}_i}$								$\delta Q_{\dot{q}_j}$
	$-\delta M_{\dot{q}_i}$					$-\delta M_{\dot{p}_j}$		
		$-\delta M_{\dot{q}_i}$					$-\delta M_{\dot{p}_j}$	

$C_0 =$

$\delta T_{q_i}$			$\delta T_{p_i}$					$\delta T_5$
	$\delta R_{q_i}$ $-\delta H_{q_i}$	$\delta H_{q_i}$ $+\delta R_{q_i}$		$\delta R_{p_i}$	$\delta H_{p_i}$	$\delta R_p$ $-\delta H_p$	$\delta H_p$ $+\delta R_p$	
	$\delta H_{q_i}$ $+\delta R_{q_i}$	$-\delta R_{q_i}$ $+\delta H_{q_i}$		$\delta H_{p_i}$	$-\delta R_{p_i}$	$\delta H_p$ $+\delta R_p$	$-\delta R_p$ $+\delta H_p$	
$\delta Q_{q_i}$			$\delta Q_{p_i}$					$\delta Q_5$
	$-\delta M_{q_i}$	$-\delta M_{q_i}$		$-\delta M_{p_i}$		$-\delta M_p$	$-\delta M_p$	
	$\delta M_{q_i}$	$-\delta M_{q_i}$			$-\delta M_{p_i}$	$\delta M_p$	$-\delta M_p$	

$$\Sigma_1 =$$

		$\delta T_\lambda$			$\delta T_\xi$
$-\delta(H_p + R_p)$	$\delta R_r$		$\delta H_{\dot{p}}$	$-\delta R_{\dot{p}}$	
$\delta R_r$	$\delta(H_p + R_p)$		$-\delta R_{\dot{p}}$	$-\delta H_{\dot{p}}$	
		$\delta Q_\lambda$			$\delta Q_\xi$
	$-\delta M_p$			$\delta M_{\dot{p}}$	
$\delta M_p$			$-\delta M_{\dot{p}}$		

$$\mathcal{Z}_0 =$$

				$-\delta_\mu T_\lambda$	$\delta T_5$
			$\delta \lambda R_r$	$\delta \lambda (H_\mu + R_\mu)$	
			$\delta \lambda (H_\mu + R_\mu)$	$-\delta \lambda R_r$	
				$-\delta_\mu Q_\lambda$	$\delta Q_5$
			$-\delta \lambda M_\mu$		
				$-\delta \lambda M_\mu$	

$$D_G =$$

$\delta \sin \alpha T_\lambda$		$-\delta \cos \alpha T_\lambda$
$\delta \cos \alpha (H_p + R_p)$	$\delta R_r$	$\delta \sin \alpha (H_p + R_p)$
$-\delta \cos \alpha R_r$	$\delta (H_p + R_p)$	$-\delta \sin \alpha R_r$
$\delta \sin \alpha Q_\lambda$		$-\delta \cos \alpha Q_\lambda$
	$-\delta M_p$	
$-\delta \cos \alpha M_p$		$-\delta \sin \alpha M_p$

### Nonaxial flow

Consider now the case of nonaxial flow,  $\mu > 0$ . This case includes helicopter mode forward flight, and conversion mode flight for the tilting proprotor aircraft. The aerodynamic coefficients are then periodic functions of  $\psi_m$ . Hence the equations of motion for the system have periodic coefficients, due to the periodically varying aerodynamics of the edgewise moving rotor.

One can express the aerodynamic coefficients as Fourier series, and then obtain the coefficients of the nonrotating equations of motion in terms of these harmonics. For the general rotor considered here, it would be necessary to evaluate the harmonics of the aerodynamic coefficients numerically, however. It is simplest therefore to just sum (numerically) the coefficients over  $m = 1 \dots N$  as is required in finding the nonrotating equations of motion and the net hub forces and moments. The nonrotating coordinates for the rotor motion (Fourier coordinate transformation) are also introduced.

For the periodic coefficient case, it is necessary to specify the number of blades  $N$ , since the periodic coefficients depend on  $N$ ; also, the periodic coefficients couple all the rotor nonrotating degrees of freedom, so more than the 0, 1C, and 1S variables are involved with the shaft motion (if  $N > 3$ ). We shall consider only the case  $N = 3$ ; then the 0, 1C, and 1S degrees of freedom are the complete set, even for the periodic coefficient case. The period of the equations in the nonrotating frame is  $\Delta\psi = 2\pi/N$ .

Again we write the aerodynamic forces in matrix form, as

$$\begin{aligned} -M_{aero} &= A_1 \dot{x}_R + A_0 x_R + \tilde{A}_1 \dot{\alpha} + \tilde{A}_0 \alpha - B_0 g \\ F_{aero} &= C_1 \dot{x}_R + C_0 x_R + \tilde{C}_1 \dot{\alpha} + \tilde{C}_0 \alpha + D_0 g \end{aligned}$$

where now the coefficients A, B, C, and D are periodic functions of  $\psi$  (period  $2\pi/N$ ). The matrices of the aerodynamic coefficients follow. The notation

$$C = \cos \psi_m$$

$$S = \sin \psi_m$$

is used ( $\psi_m = \psi + m \Delta\psi$ ). Note that each matrix is a summation over all N blades ( $N = 3$  in this case).



$$A_1 = \delta \frac{1}{2} \sum_3$$

$-M_{qkq_i}$	$-M_{qkq_i} C$		$-M_{qk\dot{p}} C$	$-M_{qk\dot{p}} S$	
	$-M_{qkq_i} S$				$-M_{qk\dot{q}}$
$-M_{qkq_i} 2C$	$-M_{qkq_i} 2C^2$		$-M_{qk\dot{p}} 2C^2$	$-M_{qk\dot{p}} 2CS$	
	$-M_{qkq_i} 2CS$				$-M_{qk\dot{q}} 2C$
$-M_{qkq_i} 2S$	$-M_{qkq_i} 2CS$		$-M_{qk\dot{p}} 2CS$	$-M_{qk\dot{p}} 2S^2$	
	$-M_{qkq_i} 2S^2$				$-M_{qk\dot{q}} 2S$
$-M_{pkq_i}$	$-M_{pkq_i} C$	$-M_{pk\dot{p}}$	$-M_{pk\dot{p}} C$	$-M_{pk\dot{p}} S$	
	$-M_{pkq_i} S$		$-M_{pk\dot{p}} S$		$-M_{pk\dot{q}}$
$-M_{pkq_i} 2C$		$-M_{pk\dot{p}} 2C$	$-M_{pk\dot{p}} 2C^2$	$-M_{pk\dot{p}} 2CS$	
	$-M_{pkq_i} 2C^2$	$-M_{pk\dot{p}} 2C^2$			$-M_{pk\dot{q}} 2C$
	$-M_{pkq_i} 2CS$		$-M_{pk\dot{p}} 2CS$		
$-M_{pkq_i} 2S$		$-M_{pk\dot{p}} 2S$	$-M_{pk\dot{p}} 2CS$	$-M_{pk\dot{p}} 2S^2$	
	$-M_{pkq_i} 2CS$	$-M_{pk\dot{p}} 2CS$			$-M_{pk\dot{q}} 2S$
	$-M_{pkq_i} 2S^2$	$-M_{pk\dot{p}} 2S^2$			
$-M_{q_i} 2C$	$-M_{q_i} 2C^2$		$-M_{\dot{p}} 2C^2$	$-M_{\dot{p}} 2CS$	
	$-M_{q_i} 2CS$				$-M_{\dot{q}} 2C$
$-M_{q_i} 2S$	$-M_{q_i} 2CS$		$-M_{\dot{p}} 2CS$	$-M_{\dot{p}} 2S^2$	
	$-M_{q_i} 2S^2$				$-M_{\dot{q}} 2S$

$$A_0 = \gamma \frac{1}{N} \sum_m$$

$-M_{qq};$	$M_{qp}; S$ $-M_{qp}; C$	$-M_{qp}; C$	$-M_{qp};$	$-M_{qp}; C$	$M_{qp}; S$ $-M_{qp}; C$	$-M_{qp}; C$	$-M_{qs}$
$-M_{qp}; 2C$	$M_{qp}; 2CS$ $-M_{qp}; 2C^2$	$-M_{qp}; 2C^2$	$-M_{qp}; 2C$	$-M_{qp}; 2C^2$	$M_{qp}; 2CS$ $-M_{qp}; 2C^2$	$-M_{qp}; 2CS$	$-M_{qs}; 2C$
$-M_{qp}; 2S$	$M_{qp}; 2S^2$ $-M_{qp}; 2CS$	$-M_{qp}; 2CS$	$-M_{qp}; 2S$	$-M_{qp}; 2CS$	$M_{qp}; 2S^2$ $-M_{qp}; 2CS$	$-M_{qp}; 2CS$	$-M_{qs}; 2S$
$-M_{pq};$	$+M_{pq}; S$ $-M_{pq}; C$	$-M_{pq}; C$	$-M_{pq};$	$-M_{pq}; C$	$+M_{pq}; S$ $-M_{pq}; C$	$-M_{pq}; C$	$-M_{ps}$
$-M_{pq}; 2C$	$+M_{pq}; 2CS$ $-M_{pq}; 2C^2$	$-M_{pq}; 2C^2$	$-M_{pq}; 2C$	$-M_{pq}; 2C^2$	$+M_{pq}; 2CS$ $-M_{pq}; 2C^2$	$-M_{pq}; 2CS$	$-M_{ps}; 2C$
$-M_{pq}; 2S$	$+M_{pq}; 2S^2$ $-M_{pq}; 2CS$	$-M_{pq}; 2CS$	$-M_{pq}; 2S$	$-M_{pq}; 2CS$	$+M_{pq}; 2S^2$ $-M_{pq}; 2CS$	$-M_{pq}; 2CS$	$-M_{ps}; 2S$
$-M_q; 2C$	$+M_q; 2CS$ $-M_q; 2C^2$	$-M_q; 2C^2$	$-M_p; 2C$	$-M_p; 2C^2$	$+M_p; 2CS$ $-M_p; 2C^2$	$-M_p; 2CS$	$-M_s; 2C$
$-M_q; 2S$	$+M_q; 2S^2$ $-M_q; 2CS$	$-M_q; 2CS$	$-M_p; 2S$	$-M_p; 2CS$	$+M_p; 2S^2$ $-M_p; 2CS$	$-M_p; 2CS$	$-M_s; 2S$

$$\tilde{A}_1 = \delta \frac{1}{N} \sum_m$$

$M_{q\mu\mu} S$	$-M_{q\mu\mu} C$		$-M_{q\mu\beta} S$	$M_{q\mu\beta} C$	
		$-M_{q\mu\lambda}$			$-M_{q\mu j}$
$M_{q\mu\mu} 2CS$	$-M_{q\mu\mu} 2C^2$		$-M_{q\mu\beta} 2CS$	$M_{q\mu\beta} 2C^2$	
		$-M_{q\mu\lambda} 2C$			$-M_{q\mu j} 2C$
$M_{q\mu\mu} 2S^2$	$-M_{q\mu\mu} 2CS$		$-M_{q\mu\beta} 2S^2$	$M_{q\mu\beta} 2CS$	
		$-M_{q\mu\lambda} 2S$			$-M_{q\mu j} 2S$
$M_{p\mu\mu} S$	$-M_{p\mu\mu} C$		$-M_{p\mu\beta} S$	$M_{p\mu\beta} C$	
		$-M_{p\mu\lambda}$			$-M_{p\mu j}$
$M_{p\mu\mu} 2CS$	$-M_{p\mu\mu} 2C^2$		$-M_{p\mu\beta} 2CS$	$M_{p\mu\beta} 2C^2$	
		$-M_{p\mu\lambda} 2C$			$-M_{p\mu j} 2C$
$M_{p\mu\mu} 2S^2$	$-M_{p\mu\mu} 2CS$		$-M_{p\mu\beta} 2S^2$	$M_{p\mu\beta} 2CS$	
		$-M_{p\mu\lambda} 2S$			$-M_{p\mu j} 2S$
$M_\mu 2CS$	$-M_\mu 2C^2$		$-M_\beta 2CS$	$M_\beta 2C^2$	
		$-M_\lambda 2C$			$-M_j 2C$
$M_\mu 2S^2$	$-M_\mu 2CS$		$-M_\beta 2S^2$	$M_\beta 2CS$	
		$-M_\lambda 2S$			$-M_j 2S$

$$\tilde{A}_0 = \delta \frac{1}{2} \sum$$

			$-\lambda M_{qkr} C$	$\mu M_{qkl}$ $-\lambda M_{qkr} S$	$-M_{qk5}$
			$-\lambda M_{qkr} 2C^2$	$\mu M_{qkl} 2C$ $-\lambda M_{qkr} 2CS$	$-M_{qk5} 2C$
			$-\lambda M_{qkr} 2CS$	$\mu M_{qkl} 2S$ $-\lambda M_{qkr} 2S^2$	$-M_{qk5} 2S$
			$-\lambda M_{prk} C$	$\mu M_{prk}$ $-\lambda M_{prk} S$	$-M_{pr5}$
			$-\lambda M_{prk} 2C^2$	$\mu M_{prk} 2C$ $-\lambda M_{prk} 2CS$	$-M_{pr5} 2C$
			$-\lambda M_{prk} 2CS$	$\mu M_{prk} 2S$ $-\lambda M_{prk} 2S^2$	$-M_{pr5} 2S$
			$-\lambda M_r 2C^2$	$\mu M_r 2C$ $-\lambda M_r 2CS$	$-M_5 2C$
			$-\lambda M_r 2CS$	$\mu M_r 2S$ $-\lambda M_r 2S^2$	$-M_5 2S$

$$B_G = \gamma \frac{1}{N} \sum_m$$

$\cos \alpha M_{K\mu} S$	$M_{K\mu} C$	$\sin \alpha M_{K\mu} S$
$\sin \alpha M_{K\lambda}$		$-\cos \alpha M_{K\lambda}$
$\cos \alpha M_{K\mu} 2CS$	$M_{K\mu} 2C^2$	$\sin \alpha M_{K\mu} 2CS$
$\sin \alpha M_{K\lambda} 2C$		$-\cos \alpha M_{K\lambda} 2C$
$\cos \alpha M_{K\mu} 2S^2$	$M_{K\mu} 2CS$	$\sin \alpha M_{K\mu} 2S^2$
$\sin \alpha M_{K\lambda} 2S$		$-\cos \alpha M_{K\lambda} 2S$
$\cos \alpha M_{P\mu} S$	$M_{P\mu} C$	$\sin \alpha M_{P\mu} S$
$\sin \alpha M_{P\lambda}$		$-\cos \alpha M_{P\lambda}$
$\cos \alpha M_{P\mu} 2CS$	$M_{P\mu} 2C^2$	$\sin \alpha M_{P\mu} 2CS$
$\sin \alpha M_{P\lambda} 2C$		$-\cos \alpha M_{P\lambda} 2C$
$\cos \alpha M_{P\mu} 2S^2$	$M_{P\mu} 2CS$	$\sin \alpha M_{P\mu} 2S^2$
$\sin \alpha M_{P\lambda} 2S$		$-\cos \alpha M_{P\lambda} 2S$
$\cos \alpha M_\mu 2CS$	$M_\mu 2C^2$	$\sin \alpha M_\mu 2CS$
$\sin \alpha M_\lambda 2C$		$-\cos \alpha M_\lambda 2C$
$\cos \alpha M_\mu 2S^2$	$M_\mu 2CS$	$\sin \alpha M_\mu 2S^2$
$\sin \alpha M_\lambda 2S$		$-\cos \alpha M_\lambda 2S$

$C_1 =$

$T_i$	$T_i C$	$T_i S$				$T_p C$	$T_p S$	$T_s$
$H_i 2S$	$H_i 2CS$	$H_i 2S^2$				$H_p 2CS$	$H_p 2S^2$	$H_s 2S$
$R_i 2C$	$R_i 2C^2$	$R_i 2CS$				$R_p 2C^2$	$R_p 2CS$	$R_s 2C$
$H_i 2C$	$H_i 2C^2$	$H_i 2CS$				$H_p 2C^2$	$H_p 2CS$	$H_s 2C$
$-R_i 2S$	$-R_i 2CS$	$-R_i 2S^2$				$-R_p 2CS$	$-R_p 2S^2$	$-R_s 2S$
$Q_i$	$Q_i C$	$Q_i S$				$Q_p C$	$Q_p S$	$Q_s$
$-M_i 2C$	$-M_i 2C^2$	$-M_i 2CS$				$-M_p 2C^2$	$-M_p 2CS$	$-M_s 2C$
$-M_i 2S$	$-M_i 2CS$	$-M_i 2S^2$				$-M_p 2CS$	$-M_p 2S^2$	$-M_s 2S$

$$C_0 = \gamma \frac{1}{N} \sum_m$$

$T_q:$	$-T_{q: S}$ $T_{q: C}$	$T_{q: C}$ $T_{q: S}$	$T_p:$	$T_{p: C}$	$T_{p: S}$	$-T_{p: S}$ $T_{p: C}$	$T_{p: C}$ $T_{p: S}$	$T_S$
$H_q: 2S$ $R_q: 2C$	$-H_{q: 2S^2}$ $H_{q: 2CS}$ $-R_{q: 2CS}$ $R_{q: 2C^2}$	$H_{q: 2CS}$ $H_{q: 2S^2}$ $R_{q: 2C^2}$ $R_{q: 2CS}$	$H_p: 2S$ $R_p: 2C$	$H_{p: 2CS}$ $R_{p: 2C^2}$	$H_{p: 2S^2}$ $R_{p: 2CS}$	$-H_{p: 2S^2}$ $H_{p: 2CS}$ $-R_{p: 2CS}$ $R_{p: 2C^2}$	$H_{p: 2CS}$ $H_{p: 2S^2}$ $R_{p: 2C^2}$ $R_{p: 2CS}$	$H_S 2S$ $R_S 2C$
$H_q: 2C$ $-R_q: 2S$	$-H_{q: 2CS}$ $H_{q: 2C^2}$ $+R_{q: 2S^2}$ $-R_{q: 2CS}$	$H_{q: 2C^2}$ $H_{q: 2CS}$ $-R_{q: 2CS}$ $-R_{q: 2S^2}$	$H_p: 2C$ $-R_p: 2S$	$H_{p: 2C^2}$ $-R_{p: 2CS}$	$H_{p: 2CS}$ $-R_{p: 2S^2}$	$-H_{p: 2CS}$ $H_{p: 2C^2}$ $+R_{p: 2S^2}$ $-R_{p: 2CS}$	$H_{p: 2C^2}$ $H_{p: 2CS}$ $-R_{p: 2CS}$ $-R_{p: 2S^2}$	$H_S 2C$ $-R_S 2S$
$Q_q:$	$-Q_{q: S}$ $Q_{q: C}$	$Q_{q: C}$ $Q_{q: S}$	$Q_p:$	$Q_{p: C}$	$Q_{p: S}$	$-Q_{p: S}$ $Q_{p: C}$	$Q_{p: C}$ $Q_{p: S}$	$Q_S$
$-M_q: 2C$	$M_{q: 2CS}$ $-M_{q: 2C^2}$	$-M_{q: 2C^2}$ $-M_{q: 2CS}$	$-M_p: 2C$	$-M_{p: 2C^2}$	$-M_{p: 2CS}$	$M_{p: 2CS}$ $-M_{p: 2C^2}$	$-M_{p: 2C^2}$ $-M_{p: 2CS}$	$-M_S 2C$
$-M_q: 2S$	$M_{q: 2S^2}$ $-M_{q: 2CS}$ $-M_{q: 2S^2}$	$-M_{q: 2CS}$ $-M_{q: 2S^2}$	$-M_p: 2S$	$-M_{p: 2CS}$	$-M_{p: 2S^2}$	$M_{p: 2S^2}$ $-M_{p: 2CS}$ $-M_{p: 2S^2}$	$-M_{p: 2CS}$ $-M_{p: 2S^2}$	$-M_S 2S$

$$\tilde{C}_i = \delta \frac{1}{2} \sum_m$$

$-T_p S$	$T_p C$	$T_\lambda$	$T_{\dot{p}} S$	$-T_{\dot{p}} C$	$T_{\dot{j}}$
$-H_p 2S^2$	$H_p 2CS$	$H_\lambda 2S$	$H_{\dot{p}} 2S^2$	$-H_{\dot{p}} 2CS$	$H_{\dot{j}} 2S$
$-R_p 2C^2$	$-R_p 2CS$	$R_\lambda 2C$	$R_{\dot{p}} 2CS$	$-R_{\dot{p}} 2C^2$	$R_{\dot{j}} 2C$
$-R_p 2CS$	$R_p 2C^2$				
$-H_p 2CS$	$H_p 2C^2$	$H_\lambda 2C$	$H_{\dot{p}} 2CS$	$-H_{\dot{p}} 2C^2$	$H_{\dot{j}} 2C$
$R_p 2CS$	$R_p 2S^2$	$-R_\lambda 2S$	$-R_{\dot{p}} 2S^2$	$R_{\dot{p}} 2CS$	$-R_{\dot{j}} 2S$
$R_p 2S^2$	$-R_p 2CS$				
$-Q_p S$	$Q_p C$	$Q_\lambda$	$Q_{\dot{p}} S$	$-Q_{\dot{p}} C$	$Q_{\dot{j}}$
$M_p 2CS$	$-M_p 2C^2$	$-M_\lambda 2C$	$-M_{\dot{p}} 2CS$	$+M_{\dot{p}} 2C^2$	$-M_{\dot{j}} 2C$
$M_p 2S^2$	$-M_p 2CS$	$-M_\lambda 2S$	$-M_{\dot{p}} 2S^2$	$M_{\dot{p}} 2CS$	$-M_{\dot{j}} 2S$



$$\lambda_0 = \gamma \frac{1}{2} \sum_m$$

			$\lambda T_\mu C$	$-\mu T_\lambda$ $\lambda T_\mu S$	$T_\lambda$ $T_\mu$
			$\lambda H_\mu 2CS$ $-\lambda R_\mu 2CS$ $\lambda R_\mu 2C^2$	$\lambda R_\mu 2C^2$ $-\mu H_\lambda 2S$ $\lambda H_\mu 2S^2$ $-\mu R_\lambda 2C$ $\lambda R_\mu 2CS$	$H_\lambda 2S$ $R_\lambda 2C$
			$\lambda H_\mu 2C^2$ $\lambda R_\mu 2S^2$ $-\lambda R_\mu 2CS$	$-\lambda R_\mu 2CS$ $-\mu H_\lambda 2C$ $\lambda H_\mu 2CS$ $\mu R_\lambda 2S$ $-\lambda R_\mu 2S^2$	$H_\lambda 2C$ $-R_\lambda 2S$
			$\lambda Q_\mu C$	$-\mu Q_\lambda$ $\lambda Q_\mu S$	$Q_\lambda$ $Q_\mu$
			$-\lambda M_\mu 2C^2$	$\mu M_\lambda 2C$ $-\lambda M_\mu 2CS$	$-M_\lambda 2C$
			$-\lambda M_\mu 2CS$	$\mu M_\lambda 2S$ $-\lambda M_\mu 2S^2$	$-M_\lambda 2S$

$$D_G = \delta \frac{1}{N} \sum_m$$

$\cos \alpha T_\mu S$	$T_\mu C$	$\sin \alpha T_\mu S$
$\sin \alpha T_\lambda$		$-\cos \alpha T_\lambda$
$\cos \alpha H_\mu 2S^2$	$H_\mu 2CS$ $R_r 2C^2$ $-R_\mu 2CS$	$\sin \alpha H_\mu 2S^2$
$\cos \alpha R_\mu 2C^2$		$\sin \alpha R_\mu 2C^2$
$\sin \alpha H_\lambda 2S$		$-\cos \alpha H_\lambda 2S$
$\cos \alpha R_r 2CS$		$\sin \alpha R_r 2CS$
$\sin \alpha R_\lambda 2C$		$-\cos \alpha R_\lambda 2C$
$\cos \alpha H_\mu 2CS$	$H_\mu 2C^2$ $R_\mu 2S^2$ $-R_r 2CS$	$\sin \alpha H_\mu 2CS$
$-\cos \alpha R_\mu 2CS$		$-\sin \alpha R_\mu 2CS$
$\sin \alpha H_\lambda 2C$		$-\cos \alpha H_\lambda 2C$
$-\cos \alpha R_r 2S^2$		$-\sin \alpha R_r 2S^2$
$-\sin \alpha R_\lambda 2S$		$\cos \alpha R_\lambda 2S$
$\cos \alpha Q_\mu S$	$Q_\mu C$	$\sin \alpha Q_\mu S$
$\sin \alpha Q_\lambda$		$-\cos \alpha Q_\lambda$
$-\cos \alpha M_\mu 2CS$	$-M_\mu 2C^2$	$-\sin \alpha M_\mu 2CS$
$-\sin \alpha M_\lambda 2C$		$\cos \alpha M_\lambda 2C$
$-\cos \alpha M_\mu 2S^2$	$-M_\mu 2CS$	$-\sin \alpha M_\mu 2S^2$
$-\sin \alpha M_\lambda 2S$		$\cos \alpha M_\lambda 2C$

### Constant Coefficient Approximation

Finally, we consider a constant coefficient approximation for the nonaxial flow case. This approximation uses the mean values of the periodic coefficients of the differential equations. A constant coefficient approximation is desirable (if it is demonstrated to be accurate enough) because the calculation required for the analysis is considerably reduced compared to the periodic coefficient equations, and because the powerful techniques for analyzing time-invariant (constant coefficient) linear differential equations are applicable. It is only an approximation to the correct dynamics however; the accuracy of the approximation must be determined by comparison with the correct periodic coefficient solutions.

To find the mean value of the coefficients, we apply the operator

$$\frac{1}{2\pi} \int_0^{2\pi} (\dots) d\psi$$

to the periodic coefficients given above. The result is terms of the form

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{N} \sum_m \begin{matrix} \cos \psi_m \\ \sin \psi_m \\ 2 \cos^2 \psi_m \\ 2 \sin^2 \psi_m \\ 2 \sin \psi_m \cos \psi_m \end{matrix} M(\psi_m) \right) d\psi \\ &= \frac{1}{N} \sum_m \frac{1}{2\pi} \int_0^{2\pi} \left( \begin{matrix} 1 \\ 0 \\ 2 \cos^2 \psi \\ 2 \sin^2 \psi \\ 2 \sin \psi \cos \psi \end{matrix} \right) M d\psi_m \\ &= \begin{pmatrix} M^0 \\ \frac{1}{2} M^{1c} \\ \frac{1}{2} M^{1s} \\ M^0 + \frac{1}{2} M^{2c} \\ M^0 - \frac{1}{2} M^{2c} \\ \frac{1}{2} M^{2s} \end{pmatrix} \end{aligned}$$

where  $M^{nc, ns}$  are the harmonics of a Fourier series representation of the rotating blade aerodynamic coefficient  $M$ :

$$M(\psi_m) = M^0 + \sum_{n=1}^{\infty} M^{nc} \cos n\psi_m + M^{ns} \sin n\psi_m$$

In the present case, these harmonics must be evaluated numerically. We evaluate  $M$  at  $J$  points, equally spaced around the azimuth:

$$M^0 = \frac{1}{J} \sum M_j$$

$$M^{nc, ns} = \frac{2}{J} \sum M_j \begin{matrix} \cos n\psi_j \\ \sin n\psi_j \end{matrix}$$

where

$$M_j = M(\psi_j)$$

$$\psi_j = j\Delta\psi, \quad j = 1 \dots J$$

$$\Delta\psi = \frac{2\pi}{J}$$

The harmonics up to the second ( $n = 2$ ) are required here. This Fourier interpolation formula requires then for good accuracy about  $J > 12$  (a  $30^\circ$  azimuth increment). Using these expressions, the required harmonics are:

$$\begin{pmatrix} M^0 \\ \frac{1}{2} M^{1c} \\ \frac{1}{2} M^{1s} \\ M^0 + \frac{1}{2} M^{2c} \\ M^0 - \frac{1}{2} M^{2c} \\ \frac{1}{2} M^{2s} \end{pmatrix} = \frac{1}{J} \sum_j \begin{pmatrix} 1 \\ \cos \psi \\ \sin \psi \\ 2 \cos^2 \psi \\ 2 \sin^2 \psi \\ 2 \sin \psi \cos \psi \end{pmatrix} M(\psi_j)$$

It follows then that the constant coefficient approximation is obtained from the periodic coefficient expressions by the simple transformation:

$$\frac{1}{N} \sum_{m=1}^N \begin{pmatrix} \frac{1}{c} \\ s \\ 2c^2 \\ 2s^2 \\ 2cs \end{pmatrix} M(\psi_m) \Rightarrow \frac{1}{J} \sum_{j=1}^J \begin{pmatrix} \frac{1}{c} \\ s \\ 2c^2 \\ 2s^2 \\ 2cs \end{pmatrix} M(\psi_j)$$

The summation over  $N$  blades ( $m = 1 \dots N$ ,  $\Delta\psi = 2\pi/N$ ) for the periodic coefficient case is replaced by a summation over the rotor azimuth ( $j = 1 \dots J$ ,  $\Delta\psi = 2\pi/J$ ) for the constant coefficient approximation. This is quite convenient, since the same procedure may be used to evaluate the coefficients for the two cases, with simply a change in the azimuth increment. The periodic coefficients must be evaluated throughout the period of  $\psi = 0$  to  $2\pi/N$  of course; while the constant coefficient approximation (the mean values only) is evaluated only once.

With the substitution  $\frac{1}{N} \sum \Rightarrow \frac{1}{J} \sum$ , the results given above for the periodic coefficient matrices are directly applicable to the constant coefficient approximation as well.

## ROTOR TRIM

There are two requirements in the dynamics analysis for the trim, equilibrium solution for the rotor blade motion and rotor performance: first, the trim bending deflection  $(x_0 t + z_0 k)$  is required for the coefficients, particularly when the blade torsion dynamics are involved; secondly, the evaluation of the aerodynamic coefficients requires the lift and drag loading of the rotor blade. The trim bending deflection is assumed to be independent of  $\mu$  in the analysis, so the mean value must be used when  $\mu > 0$ ; for the aerodynamic coefficients, the periodic variation of the trim blade aerodynamics when  $\mu > 0$  will be included however. The dynamics analysis (the evaluation of the coefficients of the equations of motion) must be preceded therefore by a preliminary calculation of the rotor equilibrium motion. The trim solution for the blade motion is periodic in the rotating frame for the general case of nonaxial flow; for  $\mu = 0$ , axial flow, the blade motion is steady in the rotating frame. For the trim blade motion solution we shall consider only the bending and gimbal degrees of freedom. It is assumed that there is no shaft motion, gusts, rotational speed perturbation, or torsion/pitch motion (except cyclic control and any bending/torsion coupling) in the trim solution.

The trim solution involves the numerical integration of the differential equations of motion for a single blade in the rotating frame, until the blade motion converges to the desired periodic solution. The equations for the blade motion are obtained from the above analysis, and are for the bending and gimbal degrees of freedom:

$$\begin{aligned} I_{q_k}^* (\ddot{q}_k + \gamma_1 v_k \dot{q}_k + v_k^2 q_k) + 2 \sum I_{q_k q_j}^* \dot{q}_j \\ = I_{q_k}^* + \gamma \int_0^1 \ddot{q}_k \cdot \left( \frac{F_2}{a^2} r_0 - \frac{F_1}{a^2} k_0 \right) dr \end{aligned}$$

$$I_0^* (v_0^2 - 1) \begin{pmatrix} \beta_{0c} \\ \beta_{0s} \end{pmatrix} = \frac{2}{J} \sum_j \begin{pmatrix} \cos \psi_j \\ \sin \psi_j \end{pmatrix} \gamma \int_0^1 \frac{F_2}{a^2} r dr$$

where the inertia constants are defined above, and the aerodynamic forces are evaluated using the trim velocity components (for which expressions are given above).

After the integration of the blade motion converges to a periodic solution, the rotor performance may be evaluated, i.e. the mean aerodynamic forces and moments the rotor produces at the hub, particularly the rotor thrust and torque coefficients. The Fourier harmonics of the blade bending motion are also evaluated. From the zeroth harmonics of the bending motion, the mean bending deflection of the blade may be evaluated.

For axial flow,  $\mu = 0$ , integration of the blade motion is not required; for the gimbal motion is zero (assuming no cyclic pitch input) and the equation for the blade bending modal deflection reduces to

$$I_{q_k}^* \partial_k^2 q_k = I_{q_k}^* + \gamma \int_0^1 \eta_k \cdot \left( \frac{F_z}{a_c} \tau_a - \frac{F_x}{a_c} \tau_a \right) dr$$

## BLADE BENDING AND TORSION MODES

### Coupled bending modes of a rotating blade

Equilibrium of the elastic, inertial, and centrifugal bending moments on the blade gives the differential equation for the coupled flap/lag bending of the rotating blade. For free vibration -- the homogeneous equation (no forcing) with harmonic motion at the natural frequency  $\omega$  -- we obtain the modal equation for bending of the blade:

$$(\mathbf{EI} \eta'')'' - \Omega^2 \left( \int_r^R \rho \mathbf{A} \eta' \right)' - m \ddot{\mathbf{x}} \cdot \eta - m \omega^2 \eta = 0$$

where  $\eta(r) = \mathbf{x} \cdot \mathbf{r} - \mathbf{x}_0 \cdot \mathbf{r}$  = bending deflection (mode shape)  
 $\mathbf{EI} = \mathbf{EI}_{xx} \mathbf{r} \mathbf{r} + \mathbf{EI}_{yy} \mathbf{k} \mathbf{k}$  = bending stiffness dyadic  
 $\Omega = \Omega \mathbf{k}$  = rotor rotational speed  
 $\omega$  = natural frequency of mode

This is an eigenvalue problem, a differential equation in  $r$  for the mode shapes  $\eta$  and the natural frequencies  $\omega$ . The equation with the appropriate boundary conditions constitutes a proper Sturm-Liouville eigenvalue problem. It follows that the solution exists: a series of modes  $\eta_i(r)$  and corresponding natural frequencies  $\omega_i$ ; where the modes are orthogonal with weight  $m$ , i.e. if  $i \neq k$  then

$$\int_0^R \eta_i \cdot \eta_k m dr = 0$$

and the frequencies satisfy the relation (an energy balance):

$$\omega^2 = \frac{\int_0^R \eta'' \mathbf{EI} \eta'' + \Omega^2 \int_r^R \rho \mathbf{A} \eta'^2 - m (\ddot{\mathbf{x}} \cdot \eta)^2 dr}{\int_0^R \eta^2 m dr}$$

The modal equation will be solved by a Galerkin method. The mode shape is expanded as a finite series in the functions  $\bar{f}_i(r)$ :

$$\eta = \sum c_i \bar{f}_i(r)$$



We require that each of the  $\vec{f}_1$  satisfy the boundary conditions on  $\vec{\eta}$ ; then the sum automatically does. Since a finite series is required for computation, this is an approximate calculation; the functions  $\vec{f}_1$  should then be chosen so that at least the lower frequency modes can be well represented, for best numerical accuracy. Substituting this series in the differential equation and operating with

$$\int_0^1 \vec{f}_k \cdot (\dots) dr$$

reduces the problem (after integration by parts and an application of the boundary conditions) to a set of algebraic equations for  $\vec{c} = [c_i]$

$$(A - \omega^2 B) \vec{c} = 0$$

where the coefficient matrices are

$$A_{ki} = \int_0^1 \left[ \vec{f}_k'' \frac{EI}{\Omega^2 R^4} \vec{f}_i'' + \int_r^1 \rho \omega^2 \vec{f}_k \cdot \vec{f}_i - m \vec{f}_k \cdot \vec{e}_2 \vec{f}_i \cdot \vec{e}_2 \right] dr$$

$$B_{ki} = \int_0^1 m \vec{f}_k \cdot \vec{f}_i dr$$

Eigenvalues of the matrix  $B^{-1}A$  are the natural frequencies  $\omega^2$  of the coupled bending vibration of the blade; and the corresponding eigenvectors  $\vec{c}$  give the mode shape  $\vec{\eta}$ . As a final step, the modes are normalized to unity at the tip:  $|\vec{\eta}(1)| = 1$ .

A convenient set of functions for  $f_1$  are the polynomials (ref 5):

$$f_n = \frac{(n+2)(n+3)}{6} r^{n+1} - \frac{n(n+3)}{3} r^{n+2} + \frac{n(n+1)}{6} r^{n+3}$$

(for a hinged blade  $f_1 = r$  is used). These polynomials satisfy the required boundary conditions, but are not orthogonal functions.

### Torsion modes of a nonrotating blade

Equilibrium of the elastic and inertial torsion moments gives the modal equation

$$(GJ \theta')' + I_0 \omega^2 \theta = 0$$

The modes are orthogonal with weight  $I_0$ ; i.e. if  $i \neq k$  then

$$\int_0^R \theta_i \theta_k I_0 dr = 0$$

and the frequencies satisfy the relation

$$\omega^2 = \frac{\int_0^R GJ \theta'^2 dr}{\int_0^R I_0 \theta^2 dr}$$

These are nonrotating modes, so the solution is independent of  $\Omega$  or  $\Theta$ . The equation is solved by a Galerkin method. Writing

$$\theta = \sum c_i f_i(r)$$

where the functions  $f_i$  satisfy the boundary conditions on  $\theta$ , and operating with  $\int_0^R \theta_k (\dots) dr$  on the differential equation, produces a set of algebraic equations for  $\vec{c} = [c_i]$ :

$$(A - \omega^2 B) \vec{c} = 0$$

where

$$A_{ki} = \int_0^R \frac{GJ}{R^2} f_k' f_i' dr$$

$$B_{ki} = \int_0^R I_0 f_k f_i dr$$

The eigenvalues of the matrix  $B^{-1}A$  give the natural frequencies of the torsion vibration, and the corresponding eigenvectors for  $\vec{c}$  give the modes. Finally, the torsion modes are normalized to unity at the tip,  $\zeta(1) = 1$ .

A convenient set of functions to use for  $f_1$  is the solution for the torsion modes of a uniform beam:

$$f_n = \sin \left[ \left( n - \frac{1}{2} \right) \pi \frac{r - r_{BA}}{1 - r_{BA}} \right]$$

These functions satisfy the boundary conditions, and will usually be close to the actual mode shapes.

## SUPPORT EQUATIONS OF MOTION: CANTILEVER WING

For the rotor support we consider a cantilever wing, with the rotor on a mast or pylon attached to the wing tip. Reference 4 discusses the cantilever wing as a representation of the tilting proprotor aircraft dynamics, and develops the equations of motion describing this support. The equations of motion for the wing, and the rotor motion produced by the wing are developed in reference 4; these results are adopted here with only two extensions: to arbitrary angle of attack of the rotor shaft with respect to the forward velocity; and the inclusion of a wing trailing-edge flap among the controls.

### Cantilever wing

The cantilever wing and pylon geometry is shown in figure 12. We consider a high aspect ratio, flexible wing, with the rotor on the tip. The wing is attached to an immovable support with cantilever root restraint. A pylon with large mass and moment of inertia is rigidly attached to the wing tip. The rotor is mounted on the pylon with the hub forward of the wing EA, with mast height  $h$ . A general pylon angle  $\delta_p$  is considered, from vertical in helicopter mode to horizontal in airplane mode. The wing motion consists of elastic bending, vertical and chordwise, and elastic torsion. There is no motion of the pylon relative to the wing tip, so the wing tip motion is transmitted directly to the hub, and hub forces and moments transmitted directly to the wing tip, through the mast of height  $h$ . The rotor and wing operate in a steady free stream of velocity  $V$ . The pylon (or mast, or rotor shaft) angle of attack  $\delta_p$  may be large, so it covers the entire range of tilting proprotor operation. The cases include:  $\delta_p$  near  $90^\circ$  for helicopter mode;  $\delta_p$  between  $0$  and  $90^\circ$  for conversion mode;  $\delta_p = 0$  for cruise mode; and  $V = 0$  is the case of hover flight.

The wing angle of attack is  $\delta_{w_2}$ , defined positive nose up; it is assumed that  $\delta_{w_2}$  is a small angle. The angle between the wing and the rotor shaft is then  $\delta_p - \delta_{w_2}$ ; it is this angle which determines

the transmission of motion and forces between the rotor and the wing. Recall that  $\alpha_{HP}$  is the angle of the rotor disk to the forward speed  $V$ ; here we use  $\delta_P$  for the shaft angle of attack, hence  $\delta_P = 90^\circ - \alpha_{HP}$ . We also consider small sweep angle  $\delta\omega_3$  (positive aft) and small dihedral angle  $\delta\omega_1$  (positive up) of the wing. A major effect of  $\delta\omega_3$  and  $\delta\omega_1$  is on the position of the effective elastic axis of the wing, hence on the effective mast height for the transmission of motion and forces between the rotor and the wing. The angles  $\delta\omega_1$ ,  $\delta\omega_2$ , and  $\delta\omega_3$  are removed from the orientation of the pylon and shaft at the wing tip. So the rotor shaft is in a vertical plane with no sweep or dihedral, parallel to  $V$  when  $\delta_P = 0$ ; and then  $\delta_P$  is the angle of attack of the shaft with respect to  $V$ , not with respect to the wing.

The wing is assumed to have a straight spar line, which is the locus of the local EA. The wing root is supported with cantilever restraint, and the rotor shaft is attached rigidly to the wing tip. The wing has no twist, constant chord  $c_w$ , length  $y_T$  from root to tip (semispan), with the distance  $y_w$  measured from the root, along the wing spar. The shaft length (mast height) is  $h$ , the distance the rotor hub is forward of the wing tip EA. The wing spar is roughly perpendicular to  $V$ , with small wing sweep, dihedral, and angle of attack considered. The wing root is attached to a plane defined by the forward velocity  $V$  and the vertical; then the three rotation angles  $\delta\omega_1$ ,  $\delta\omega_2$ , and  $\delta\omega_3$  define the orientation of the spar with respect to the free stream velocity. Next the pylon is rotated by  $-\delta\omega_1$ ,  $-\delta\omega_2$ , and  $-\delta\omega_3$  with respect to the wing tip, to keep the shaft parallel to  $V$ ; finally the pylon is rotated by  $\delta_P$  with respect to  $V$ , defining the orientation of the rotor.

Swept wings are usually built with a center box structure in the fuselage, where the spars are unswept, and only the wing structure outside the fuselage has swept spars. The wing is restrained at several points where the wing box is tied to the fuselage structure, or in this case to the cantilever wing fixed support. There exists an effective elastic axis for vertical bending of the wing tip: some point on the shaft or its

extension where the application of a vertical force results in purely vertical displacement of the shaft, with no rotation in pitch. Without sweep this point would be just at the wing tip EA; but with sweep a force there will produce a pitch motion of the shaft also, hence the effective EA is some distance from the wing tip EA. The effective elastic axis for the tip lies between the actual wing tip EA and the extension of the unswept spar line, the actual position depending on the degree of root restraint and sweep, and other structural details. Figure 13 illustrates the geometry involved. Reference 4 develops an elementary model for the wing bending and torsion including the shift of the effective EA due to sweep (and a similar effect due to dihedral), which is adopted here. The effective EA position is described by (figure 13):

$$\begin{aligned} h &= \text{mast height, distance hub forward wing tip EA.} \\ h_{EA} &= \text{effective mast height, distance hub forward effective EA.} \\ z_{EA} &= \text{distance hub below effective EA due to dihedral.} \end{aligned}$$

Further discussion of this effect, including the estimation of the parameters involved, is given in reference 4.

The aircraft has two contrarotating rotors, one on each wing tip. The direction of rotation of the rotor on the right wing (as in figure 12) may be either clockwise or counterclockwise. The influence of the rotor rotational direction is a few signs in the equations of motion, reflecting how the rotor hub forces and moments excite the wing motion, and how the wing produces motion of the rotor shaft. As in reference 4, the notation  $\Omega$  is used to carry this influence of the rotor rotation direction, where  $\Omega$  takes only the values  $\pm 1$ :

$$\Omega = \begin{cases} +1, & \text{rotor rotation clockwise on right wing, counterclockwise on left.} \\ -1, & \text{rotor rotation counterclockwise on right wing, clockwise on left.} \end{cases}$$

### Wing Motion

The wing motion is described by elastic bending and torsion of the spar; the pylon, and with it the rotor shaft, is rigidly attached to the wing tip. Elastic bending results in deflection of the wing spar with components both perpendicular to the wing surface (vertical or beam bending), and parallel to the wing surface (chordwise bending). Vertical and chordwise bending are defined with respect to the direction of the local principle axes of the section. There is no wing twist, so these principle axes are the same all along the span, but they are not vertical and horizontal axes because of the wing sweep, dihedral, and angle of attack. We define (figure 12) the wing bending and torsion deflection as follows:

$z_w(y_w)$  = elastic bending vertical displacement of the spar, normal to the wing surface, positive up.

$x_w(y_w)$  = elastic bending chordwise displacement of the spar, in the plane of the wing, positive rearward.

$\Theta_w(y_w)$  = pitch change of local wing section, due to elastic torsion about the local EA, positive nose up.

A modal description of the wing elastic deformation is used, and only the lowest frequency modes retained. We consider just three degrees of freedom for the wing: first mode vertical bending, chordwise bending, and torsion. The degrees of freedom representing the wing motion are:

$q_{w1}$  = wing vertical or beamwise bending, positive upward;  $q_{w1} = z_w/y_{Tw}$  at the wing tip.

$q_{w2}$  = wing chordwise bending, positive rearward;  $q_{w2} = x_w/y_{Tw}$  at the wing tip.

$p_w$  = wing elastic torsion, positive nose up;  $p_w = \Theta_w$  at the wing tip.

Associated with these degrees of freedom are mode shapes,  $\{z_w(y_w)\}$  for torsion, and  $\{x_w(y_w)\}$  for bending. These modes are normalized to 1 and to  $y_{Tw}$  respectively at the wing tip ( $y_w = y_{Tw}$ ).

From the results of reference 4, generalized to arbitrary pylon angle of attack  $\delta p$ , the rotor hub motion due to the wing degrees of freedom is:

$$\begin{bmatrix} x_h \\ y_h \\ z_h \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} = \begin{bmatrix} \gamma C & \gamma S & l \\ -l\gamma\delta_3 & +l\gamma\delta_1 & +(l_{EA}-l)C \\ -l\gamma S & l\gamma C & -l\delta_1 C \\ -\gamma\delta_1 & -\gamma\delta_3 & -l\delta_3 S \\ \gamma S & -\gamma C & (l_{EA}-l)S \\ \gamma S & -\gamma C & +z_{EA} C \\ \gamma S & -\gamma C & \delta_1 C \\ -\gamma\delta_3 & \gamma\delta_1 & +\delta_3 S \\ -\gamma C & -\gamma S & 1 \\ -\gamma C & -\gamma S & \delta_1 S \\ & & -\delta_3 C \end{bmatrix} \begin{bmatrix} q_{w1} \\ q_{w2} \\ p_w \end{bmatrix}$$

where we have written

C for  $\cos(\delta p - \delta w_2)$

S for  $\sin(\delta p - \delta w_2)$

$\gamma$  for  $\Omega \gamma_w^0 (\gamma_{TW})$

y for  $y_{T_w}$

$\delta_1$  for  $\Omega \delta w_1$

$\delta_3$  for  $\Omega \delta w_3$



# Wing Equations of Motion

From reference 4, the equation of motion for the  $q_{w1}$ ,  $q_{w2}$ , and  $p_w$  degrees of freedom of the cantilever wing, excited by the forces and moments at the rotor hub and by the wing aerodynamic forces, are:

$$\begin{bmatrix} I_{q_w}^* + m_p^* & 0 & S_w^* \\ 0 & I_{q_w}^* + I_{p_x}^* \gamma_w^2 + m_p^* & -S_w^* \delta w_2 \\ S_w^* & -S_w^* \delta w_2 & I_{p_w}^* + I_{p_y}^* \end{bmatrix} \begin{pmatrix} q_{w1} \\ q_{w2} \\ p_w \end{pmatrix} + \begin{bmatrix} C_{q1}^* & 0 & 0 \\ 0 & C_{q2}^* & 0 \\ 0 & 0 & C_p^* \end{bmatrix} \begin{pmatrix} q_{w1} \\ q_{w2} \\ p_w \end{pmatrix} + \begin{bmatrix} k_{q1}^* & 0 & 0 \\ 0 & k_{q2}^* & 0 \\ C_{p1}^* \gamma \delta \frac{\gamma_T}{\sigma_a} C & C_{p2}^* \gamma \delta \frac{\gamma_T}{\sigma_a} S & k_p^* \end{bmatrix} \begin{pmatrix} q_{w1} \\ q_{w2} \\ p_w \end{pmatrix} = \gamma \begin{pmatrix} M_{q1,aero} \\ M_{q2,aero} \\ M_{p_w,aero} \end{pmatrix}$$

$$+ \begin{bmatrix} 2\gamma S & \gamma C & h\gamma S & 2\gamma C & -\gamma \delta_3 & -\gamma S \\ -2\gamma C & \gamma S & -h\gamma C & 2\gamma S & \gamma \delta_1 & \gamma C \\ 2(l_{EA}-h)S & h & h\delta_3 S & 2\delta_3 C & 1 & -\delta_3 S \\ +2z_{EA}C & +(h_{EA}-h)C & +h\delta_1 C & -2\delta_1 S & & -\delta_1 C \end{bmatrix} \gamma \begin{bmatrix} M_{q1,aero} \\ M_{q2,aero} \\ M_{p_w,aero} \end{bmatrix}$$

The wing equations are normalized by dividing by  $(N/2)I_b$ , so the rotor exciting forces are in helicopter coefficient form. The inertias are:

$$I_{q_w}^* = \frac{1}{2I_b} \int_0^{y_{rw}} m_w \eta_w^2 dy_w$$

$$I_{p_w}^* = \frac{1}{2I_b} \int_0^{y_{rw}} I_{\theta_w} \xi_w^2 dy_w$$

$$m_p^* = \frac{m_p y_{rw}^2}{2I_b}$$

$$I_{p_x}^* = \frac{I_{p_x}}{2I_b}$$

$$I_{p_y}^* = \frac{I_{p_y}}{2I_b}$$

$$S_w^* = m_p^* \frac{EPEA}{y_{rw}}$$

where  $m_w$  is the wing mass per unit length;  $I_{\theta_w}$  is the wing section moment of inertia in pitch;  $m_p$  is the pylon mass (without the rotor);  $I_{p_x}$  and  $I_{p_y}$  are the pylon yaw and pitch moments of inertia, without the rotor, about the wing tip effective EA; and  $z_{pEA}$  is the distance the pylon CG (without the rotor) is ahead of the wing tip effective EA. For the proprotor configuration, the pylon mass is so large that it dominates the wing inertias. Hence the inertia is primarily that of the pylon and rotor, with the wing contributing elastic restraint of the motion. The wing structural spring constants are  $K_{q_1}^*$ ,  $K_{q_2}^*$ , and  $K_p^*$ ; these were evaluated by matching the predicted frequencies of the wing modes to the values obtained experimentally.  $C_{q_1}^*$ ,  $C_{q_2}^*$ , and  $C_p^*$  are the structural damping constants for the wing modes. Vertical bending elevates the rotor trim thrust above the inboard sections, and so gives a nose down pitch moment with effectiveness given by  $C_{pq}^*$ :

$$C_{pq}^* = \int_0^{y_{rw}} \xi_w \eta_w^2 dy_w / y_{rw} \approx \frac{2}{3}$$

Dimensionally, the spring and damping constants are

$$K^* = K / \frac{\pi}{2} I_b \Omega^2$$

$$C^* = C / \frac{\pi}{2} I_b |\Omega|$$

Hence the relative spring and damping rates vary with the rotor rotational speed; i.e. the wing frequency is really a fixed dimensional value (Hz), so the per-rev values vary with  $\Omega$ .

Additional discussion and details of the wing equations of motion are given in reference 4.

#### Wing Aerodynamics

The wing aerodynamic forces exciting bending and torsion motion of the wing are:

$$M_{q_{1,aero}} = \frac{1}{8 \frac{\pi}{2} I_b} \int_0^{y_w} F_{z_w} \gamma_w dy_w$$

$$M_{q_{2,aero}} = \frac{1}{8 \frac{\pi}{2} I_b} \int_0^{y_w} F_{x_w} \gamma_w dy_w$$

$$M_{p_{3,aero}} = \frac{1}{8 \frac{\pi}{2} I_b} \int_0^{y_w} M_w \gamma_w dy_w$$

where  $F_{z_w}$  and  $F_{x_w}$  are the vertical and chordwise aerodynamic forces on the wing section<sup>w</sup> (lift and profile plus induced drag);  $M_w$  is the aerodynamic moment about the local EA. The velocity seen by the section has perturbations due to the wing degrees of freedom, and due to aerodynamic gusts. Aerodynamic interference between the rotor and the wing is neglected. From the velocity perturbations, the perturbations of the section forces may be found, and hence the wing aerodynamic coefficients. The derivation of the wing aerodynamic coefficients follows the standard techniques of strip theory in aeroelasticity; more details of the derivation are given in reference 4. We also include here the aerodynamic force due to the deflection of a control surface (flap or aileron) on the wing trailing-edge. The geometry is shown in figure 14. A constant chord ( $c_F$ ) trailing-edge flap, extending from  $y_w = y_{F_I}$  to  $y_w = y_{F_0}$  is considered. The flap deflection angle is  $\delta_f$ , positive downward. So

$\delta_f$  is a control variable, in addition to the rotor cyclic and collective pitch controls. The result for the wing aerodynamic forces is:

$$\begin{pmatrix} M_{q_{u_1} \text{ aero}} \\ M_{q_{u_2} \text{ aero}} \\ M_{p_w \text{ aero}} \end{pmatrix} = \begin{bmatrix} C_{q_1 \dot{q}_1} & C_{q_1 \dot{q}_2} & C_{q_1 \dot{p}} \\ C_{q_2 \dot{q}_1} & C_{q_2 \dot{q}_2} & C_{q_2 \dot{p}} \\ C_{p \dot{q}_1} & C_{p \dot{q}_2} & C_{p \dot{p}} \end{bmatrix} \begin{pmatrix} q_{u_1} \\ q_{u_2} \\ p_w \end{pmatrix} \\
 + \begin{bmatrix} C_{q_1 q_1} & C_{q_1 q_2} & C_{q_1 p} \\ C_{q_2 q_1} & C_{q_2 q_2} & C_{q_2 p} \\ C_{p q_1} & C_{p q_2} & C_{p p} \end{bmatrix} \begin{pmatrix} q_{u_1} \\ q_{u_2} \\ p_w \end{pmatrix} \\
 + \begin{bmatrix} C_{q_1 u} & C_{q_1 v} & C_{q_1 w} \\ C_{q_2 u} & C_{q_2 v} & C_{q_2 w} \\ C_{p u} & C_{p v} & C_{p w} \end{bmatrix} \begin{pmatrix} u_G \\ v_G \\ w_G \end{pmatrix} \\
 + \begin{bmatrix} C_{q_1 \delta} \\ C_{q_2 \delta} \\ C_{p \delta} \end{bmatrix} \delta_f$$

The aerodynamic coefficients are:

$$C_{q_1 u} = \partial_{12} V^2 C_{L\alpha} e_1$$

$$C_{q_1 w} = \partial_{12} V C_{L\alpha} e_1$$

$$C_{q_1 v} = \delta_{u1} C_{q_1 w} + \delta_{u3} C_{q_1 u}$$

$$C_{q_1 \dot{q}_1} = -\partial_{13} V C_{L\alpha} e_2$$

$$C_{q_1 \dot{q}_2} = -\delta_{13} V C_{L0} e_2$$

$$C_{q_1 q_1} = -\delta_{12} V^2 \delta u_3 C_{L\alpha} e_3$$

$$C_{q_1 q_2} = -\delta_{12} V^2 \delta u_3 C_{L0} e_3$$

$$C_{q_1 \dot{p}} = \delta_{22} \frac{1}{2} V \left( \frac{2}{4} + \frac{\chi_{\lambda u}}{c_w} \right) C_{L\alpha} e_4$$

$$C_{q_1 p} = \delta_{12} V^2 C_{L\alpha} e_4$$

$$C_{q_1 \delta} = \delta_{12} V^2 C_{L\alpha} C_{\delta\delta}^* e_5$$

$$C_{q_2 u} = \delta_{12} V^2 (C_{D0} - \delta u_2 C_{L0}) e_1$$

$$C_{q_2 w} = \delta_{12} V (C_{D\alpha} - 2C_{L0}) e_1$$

$$C_{q_2 v} = \delta u_1 C_{q_2 w} + \delta u_3 C_{q_2 u}$$

$$C_{q_2 \dot{q}_1} = -\delta_{13} V (C_{D\alpha} - 2C_{L0}) e_2$$

$$C_{q_2 \dot{q}_2} = -\delta_{13} V (2C_{D0} - \delta u_2 C_{D\alpha}) e_2$$

$$C_{q_2 q_1} = -\delta_{12} V^2 \delta u_3 (C_{D\alpha} - 2C_{L0}) e_3$$

$$C_{q_2 q_2} = -\delta_{12} V^2 \delta u_3 (2C_{D0} - \delta u_2 C_{D\alpha}) e_3$$

$$C_{q_2 \dot{p}} = \delta_{22} \frac{1}{2} V \left[ \left( \frac{1}{2} + \frac{\chi_{\lambda u}}{c_w} \right) (C_{D\alpha} - 2C_{L0}) - \frac{1}{4} C_{L0} \right] e_4$$

$$C_{q_2 p} = \delta_{12} V^2 (C_{D\alpha} - C_{L0}) e_4$$

$$C_{q_2 \delta} = \delta_{12} V^2 (C_{D\delta} + (C_{D\alpha} - C_{L0}) C_{\delta\delta}^*) e_5$$

$$C_{p u} = \delta_{21} V 2 C_{max} f_1$$

$$C_{p w} = -\delta_{21} V \frac{\chi_{\lambda u}}{c_w} C_{L\alpha} f_1$$

$$C_{p v} = \delta u_1 C_{p w} + \delta u_3 C_{p u}$$

$$C_{p \dot{q}_1} = \delta_{22} V \frac{\chi_{\lambda u}}{c_w} C_{L\alpha} e_4$$

$$C_{Pq2} = -\delta_{22} V^2 C_{mac} e_4$$

$$C_{Pq1} = \delta_{12} V^2 C_{mac} f_2$$

$$C_{Pq2} = -\delta_{12} V^2 C_{Lo} f_2$$

$$C_{Pi} = -\delta_{31} \frac{1}{2} V \left( \frac{1}{4} + \frac{1}{2} \frac{x_{Aw}}{c_w} \right) C_{Lo} f_3$$

$$C_{PP} = -\delta_{21} V^2 \frac{x_{Aw}}{c_w} C_{Lo} f_3$$

$$C_{P\delta} = -\delta_{21} V^2 \left( \frac{x_{Aw}}{c_w} C_{\delta\delta}^* - C_{m\delta}^* \right) C_{Lo} f_4$$

$C_L$  and  $C_D$  are the aircraft trim lift and drag (profile plus induced) coefficients; and  $C_{L\alpha}$  and  $C_{D\alpha}$  their derivatives with respect to  $\alpha_w$ . The section moment characteristics are given by  $x_A$ , the distance the wing AC is behind the EA, and  $c_{m_{ac}}$ , the nose up moment coefficient about the AC. The constant

$$\delta_{nm} = \frac{C_w'' \gamma_{rw}''}{\pi r a}$$

accounts for the difference in the normalization of the wing and rotor coefficients. The constants  $e_n$  and  $f_n$  are integrals of the wing mode shapes, accounting for the way the motion produces forces on the wing:

$$e_1 = \int_0^{\gamma_{rw}} \gamma_w d\gamma_w / \gamma_{rw}^2 \approx \frac{1}{3}$$

$$e_2 = \int_0^{\gamma_{rw}} \gamma_w^2 d\gamma_w / \gamma_{rw}^3 \approx \frac{1}{5}$$

$$e_3 = \int_0^{\gamma_{rw}} \gamma_w \gamma_w' d\gamma_w / \gamma_{rw}^2 \approx \frac{1}{2}$$

$$e_4 = \int_0^{\gamma_{rw}} \gamma_w \gamma_w'' d\gamma_w / \gamma_{rw} \approx \frac{1}{4}$$

$$e_5 = \int_{\gamma_{r2}}^{\gamma_{r0}} \gamma_w d\gamma_w / \gamma_{rw} \approx \frac{1}{3} [(\gamma_{r0}/\gamma_{rw})^2 - (\gamma_{r2}/\gamma_{rw})^2]$$

$$f_1 = \int_0^{\gamma_{rw}} \gamma_w d\gamma_w / \gamma_{rw} \approx \frac{1}{2}$$

$$f_2 = \int_0^{\gamma_{rw}} \gamma_w \gamma_w' \frac{1}{2} (\gamma_{rw} - \gamma_w)^2 d\gamma_w / \gamma_{rw} \approx \frac{1}{12}$$

$$f_3 = \int_0^{\gamma_{rw}} \gamma_w^2 d\gamma_w / \gamma_{rw} \approx \frac{1}{3}$$

$$f_4 = \int_{\gamma_{r2}}^{\gamma_{r0}} \gamma_w d\gamma_w / \gamma_{rw} \approx \frac{1}{2} [(\gamma_{r0}/\gamma_{rw})^2 - (\gamma_{r2}/\gamma_{rw})^2]$$

For the flaperon coefficients, we use:

$$C_{L\delta} = \frac{1}{2\pi} \frac{\partial C_L}{\partial \delta_f} = \frac{1}{\pi} (\sqrt{1-c^2} + c\sqrt{1-c^2}) (.95 + .05 \frac{C_f}{C_w}) (1 + \frac{1-C_f/C_w}{2R_w})$$

$$C_{m\delta} = \frac{1}{2\pi} \frac{\partial C_m}{\partial \delta_f} = -\frac{1}{4\pi} ((1+c)\sqrt{1-c^2}) (.95 + .05 \frac{C_f}{C_w}) (1 + \frac{1-C_f/C_w}{2R_w})$$

$$C_{D\delta} = \frac{\partial C_D}{\partial \delta_f} \approx .02 \text{ to } .06$$

where 
$$c = 1 - 2 \frac{C_f}{C_w}$$

The first factor in these expressions is the two-dimensional thin airfoil theory result for the lift and moment due to control surface deflection; and the last two factors are corrections for the wing aspect ratio, thickness, and real flow effects on the flap effectiveness (based on ref. 6).

### $\dot{\psi}_s$ Equation of Motion

The rotational speed degree of freedom ( $\dot{\psi}_s$ ) is an important factor in the dynamics, especially with a windmilling rotor. Usually the  $\dot{\psi}_s$  equation of motion will involve the engine, drive train, interconnect shaft, and governor dynamics; here we shall consider only two limiting cases.

The first case is windmilling or autorotation operation of the rotor. The rotor is free to turn on the shaft, so no torque moments are transmitted from the rotor to the shaft, and no pylon roll motion transmitted to the rotor. Both effects are accomplished by using  $C_Q = 0$  as the equation of motion for  $\dot{\psi}_s$ . There is no spring term on  $\dot{\psi}_s$ , so the degree of freedom is really  $\ddot{\psi}_s$ , the rotor speed perturbation. It should be noted that  $\dot{\psi}_s$  is defined with respect to the pylon, which has a roll angle  $\alpha_s$ ; so the rotor speed perturbation with respect to space is the sum  $\dot{\psi}_s + \dot{\alpha}_s$ .

The second case considered here is powered operation of the rotor. It is assumed that the rotor hub rotational speed is fixed, at  $\Omega$ , with no perturbations. This case may be viewed as the limit of operation with a perfect governor on engine or rotor speed. The powered case is treated by dropping the  $\psi_s$  degree of freedom and equation; i.e. the solution is just  $\psi_s = 0$ .

Hence we add to the support equations of motion the equation for  $\dot{\psi}_s$ :

$$\frac{c\theta}{\tau_a} = 0$$

For the powered case this equation and the  $\psi_s$  degree of freedom are dropped from the system (a row and column eliminated from the matrices). For the windmilling case they are retained; note that the  $\psi_s$  equation is first order, since there is no spring term.

Reference 4 gives a further discussion of these two cases, windmilling and powered operation, and their effects on the proprotor dynamics.

#### Support Equations of Motion

We have obtained now the shaft motion and support equations of motion, which in matrix form are:

$$\alpha = C x_w$$

$$a_2 \ddot{x}_w + a_1 \dot{x}_w + a_0 x_w = b v_w + b_g g + \tilde{a} F$$

where the wing degrees of freedom ( $\vec{x}_w$ ) and the wing flap control ( $\vec{v}_w$ ) are



$$\vec{x}_w = \begin{bmatrix} q_{w1} \\ q_{w2} \\ p_w \end{bmatrix}$$

$$\vec{v}_w = [\delta_f]$$

and as defined above, the rotor hub forces and moments ( $\vec{F}$ ), shaft motion ( $\vec{\alpha}$ ), and aerodynamic gust ( $\vec{g}$ ) are:

$$\vec{F} = \begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \\ -\frac{\partial M}{\partial x} \\ \frac{\partial M}{\partial y} \\ \frac{\partial M}{\partial z} \\ -\frac{\partial N}{\partial x} \end{bmatrix}$$

$$\vec{\alpha} = \begin{bmatrix} x_a \\ y_a \\ z_a \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

$$\vec{g} = \begin{bmatrix} u_g \\ v_g \\ w_g \end{bmatrix}$$

The matrices of the coefficients of the equations of motion follow. The matrix  $c$ , relating the rotor shaft motion to the wing motion, has been given above.

$$a_2 =$$

$I_{q_w}^* + m_p^*$		$S_w^*$
	$I_{q_w}^* + I_{p_1}^* \eta^2 + m_p^*$	$-S_w^* \delta u_2$
$S_w^*$	$-S_w^* \delta u_2$	$I_{p_w}^* + I_{p_j}^*$

$C_{q_1}^*$		
$-\partial C_{q_1} \dot{q}_1$	$-\partial C_{q_1} \dot{q}_2$	$-\partial C_{q_1} \dot{p}$
	$C_{q_2}^*$	
$-\partial C_{q_2} \dot{q}_1$	$-\partial C_{q_2} \dot{q}_2$	$-\partial C_{q_2} \dot{p}$
		$C_p^*$
$-\partial C_p \dot{q}_1$	$-\partial C_p \dot{q}_2$	$-\partial C_p \dot{p}$

$$a_0 =$$

$k_{q_1}^*$		
$-\delta C_{q_1 q_1}$	$-\delta C_{q_1 q_2}$	$-\delta C_{q_1 p}$
	$k_{q_2}^*$	
$-\delta C_{q_2 q_1}$	$-\delta C_{q_2 q_2}$	$-\delta C_{q_2 p}$
$c_{q_1 y}^* \delta \frac{\partial \psi}{\partial y} C$	$c_{q_2 y}^* \delta \frac{\partial \psi}{\partial y} S$	$k_p^*$
$-\delta C_{p q_1}$	$-\delta C_{p q_2}$	$-\delta C_{p p}$

$$b =$$

$\delta C_{q_1 \delta}$
$\delta C_{q_2 \delta}$
$\delta C_{p \delta}$

$$b_G = \begin{bmatrix} \delta C_{q_1 u} & \delta C_{q_1 v} & \delta C_{q_1 w} \\ \delta C_{q_2 u} & \delta C_{q_2 v} & \delta C_{q_2 w} \\ \delta C_{p u} & \delta C_{p v} & \delta C_{p w} \end{bmatrix}$$

$$\hat{Q} =$$

$$\begin{bmatrix} & & & -1 & & \\ 2\gamma S & \gamma C & 2\gamma S & 2\gamma C & -\gamma \delta_3 & -\gamma S \\ -2\gamma C & \gamma S & -2\gamma C & 2\gamma S & \gamma \delta_1 & \gamma C \\ 2(l_{E1}-h)S & h & 2\delta_3 S & 2\delta_3 C & 1 & -\delta_3 S \\ +2\tau_{E1} C & +(l_{E1}-h)C & +h\delta_1 C & -2\delta_1 S & & -\delta_1 C \\ & -\tau_{E1} S & & & & \end{bmatrix}$$

# EQUATIONS OF MOTION

The complete set of equations of motion describing the prop-rotor and cantilever wing system may now be obtained, by substituting for the shaft motion into the rotor forces and moments, and then for the rotor forces into the wing equations. The result is a set of linear differential equations, of the form:

$$A_2 \ddot{x} + A_1 \dot{x} + A_0 x = B_v \dot{v} + B_G g$$

where the degrees of freedom (state) vector ( $\vec{x}$ ) and the input vector ( $\vec{v}$ ) are:

$$x = \begin{bmatrix} x_R \\ x_w \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_{1c} \\ \beta_{1s} \\ \theta_0 \\ \theta_{1c} \\ \theta_{1s} \\ \beta_{0c} \\ \beta_{0s} \\ \psi_3 \\ q_{w1} \\ q_{w2} \\ p_w \end{bmatrix}$$

$$v = \begin{bmatrix} v_R \\ v_w \end{bmatrix} = \begin{bmatrix} \theta_0^{can} \\ \theta_{1c}^{can} \\ \theta_{1s}^{can} \\ \delta_5 \end{bmatrix}$$

Recalling the equations for the rotor equations of motion, the rotor hub forces and moments, the shaft motion, and the wing equations of motion:

$$A_2 \ddot{x}_R + A_1 \dot{x}_R + A_0 x_R + \tilde{A}_2 \ddot{\alpha} + \tilde{A}_1 \dot{\alpha} + \tilde{A}_0 \alpha = B v_R + B_G g$$

$$F = C_2 \ddot{x}_R + C_1 \dot{x}_R + C_0 x_R + \tilde{C}_2 \ddot{\alpha} + \tilde{C}_1 \dot{\alpha} + \tilde{C}_0 \alpha + D_G g$$

$$0 = c x_w$$

$$a_2 \ddot{x}_w + a_1 \dot{x}_w + a_0 x_w = b v_w + b_G g + \tilde{a} F$$

the coefficient matrices of the complete equations of motion may be identified, as:

$$A_2 = \begin{bmatrix} A_2 & \tilde{A}_2 c \\ -\tilde{a} C_2 & a_2 - \tilde{a} \tilde{C}_2 c \end{bmatrix}$$

$$A_1 = \begin{bmatrix} A_1 & \tilde{A}_1 c \\ -\tilde{a} C_1 & a_1 - \tilde{a} \tilde{C}_1 c \end{bmatrix}$$

$$A_0 = \begin{bmatrix} A_0 & \tilde{A}_0 c \\ -\tilde{a} C_0 & a_0 - \tilde{a} \tilde{C}_0 c \end{bmatrix}$$

$$B = \begin{bmatrix} B & 0 \\ 0 & b \end{bmatrix}$$

$$B_G = \begin{bmatrix} B_G \\ b_G + \tilde{a} D_G \end{bmatrix}$$

### Treatment of Rotor Pitch/Torsion

The equations of motion have been set up including the rotor pitch and torsion degrees of freedom,  $\Theta^{(i)}$ , and with  $\Theta^{com}$  (the commanded pitch angle) as the rotor control variable. One may not wish to include these degrees of freedom in the system dynamics, but it is not possible to simply drop them at this stage. The pitch control and bending/gimbal feedback enters the system through the rigid pitch degree of freedom ( $p_0$ ), so it is necessary to first operate on the columns of the equation matrices to account for these effects. Then the degrees of freedom and equations (columns and rows of the matrices) may be dropped as appropriate. We shall consider three options for the treatment of the rotor pitch/torsion motion.

The first option is to include the pitch and torsion degrees of freedom in the system; then the equations are used as derived.

The second option is the case of a rigid control system. It is the limit of infinite control system and blade torsion stiffness. Thus the rotor blade elastic torsion motion is zero, and the response of the rigid pitch motion reduces to

$$p_0 = \Theta^{com} - \sum K_{P_i} q_i - K_{P_G} \beta_G$$

$$\text{or} \quad \begin{pmatrix} \Theta_0 \\ \Theta_{1c} \\ \Theta_{1s} \end{pmatrix}_0 = \begin{pmatrix} \Theta_0 \\ \Theta_{1c} \\ \Theta_{1s} \end{pmatrix}_{com} - \sum K_{P_i} \begin{pmatrix} \beta_0 \\ \beta_{1c} \\ \beta_{1s} \end{pmatrix}_i - K_{P_G} \begin{pmatrix} 0 \\ \beta_{Gc} \\ \beta_{Gs} \end{pmatrix}$$

Thus we operate on the columns of the  $A_0$  matrix as follows:

- subtract  $K_{P_1}$  times the  $\Theta_0^{(0)}$  column from the  $\beta_0^{(i)}$  column
- subtract  $K_{P_1}$  times the  $\Theta_{1c}^{(0)}$  column from the  $\beta_{1c}^{(i)}$  column
- subtract  $K_{P_1}$  times the  $\Theta_{1s}^{(0)}$  column from the  $\beta_{1s}^{(i)}$  column
- subtract  $K_{P_G}$  times the  $\Theta_{1c}^{(0)}$  column from the  $\beta_{Gc}$  column
- subtract  $K_{P_G}$  times the  $\Theta_{1s}^{(0)}$  column from the  $\beta_{Gs}$  column

and reconstruct the control matrix B as follows:

replace the  $\Theta_o^{con}$  column of B with minus the  $\Theta_o^{(o)}$  column of  $A_o$

replace the  $\Theta_{1c}^{con}$  column of B with minus the  $\Theta_{1c}^{(o)}$  column of  $A_o$

replace the  $\Theta_{1s}^{con}$  column of B with minus the  $\Theta_{1s}^{(o)}$  column of  $A_o$

Then the rigid pitch degrees of freedom and equations of motion are dropped from the system. Note that the above transformation is only the result of infinite control system stiffness; it would be possible to retain the elastic torsion degrees of freedom, dropping only the rigid pitch  $p_o$ .

The third option is a quasistatic approximation for the effect of the blade torsion and pitch motion. We shall neglect the acceleration and velocity terms in the torsion/pitch equations. The torsion/pitch equations then become just a static substitution relation for  $\Theta$  in the other equations of motion. This treatment retains all the static coupling effects in the  $A_o$  matrix. The required transformation of the equations is accomplished as follows. First the  $A_o$ , B, and  $B_G$  matrices are partitioned, to separate the  $\Theta$  variables and equations from the rest. Assuming the  $\Theta$  block is in the middle of  $\vec{x}$ , the state equations take the form:

$$A_2 \ddot{\vec{x}} + A_1 \dot{\vec{x}} + \begin{bmatrix} A_o^{11} & A_o^{12} & A_o^{13} \\ A_o^{21} & A_o^{22} & A_o^{23} \\ A_o^{31} & A_o^{32} & A_o^{33} \end{bmatrix} \vec{x} = \begin{bmatrix} B^1 \\ B^2 \\ B^3 \end{bmatrix} v + \begin{bmatrix} B_G^1 \\ B_G^2 \\ B_G^3 \end{bmatrix} g$$

Now the acceleration and velocity terms are dropped from the pitch equations; and we write  $\vec{x}$  still for the state variable vector, but now with the pitch degrees of freedom dropped. Hence

$$\Theta = (A_o^{22})^{-1} \left[ - \begin{bmatrix} A_o^{21} & A_o^{23} \end{bmatrix} \vec{x} + B^2 v + B_G^2 g \right]$$

which may be substituted into the remaining equations, eliminating  $\Theta$  from  $A_o$  (the pitch acceleration and velocity terms in the remaining equations



are dropped). Thus the quasistatic torsion approximation gives the following equations of motion, in terms of the reduced state variable  $\vec{x}$  (without the torsion/pitch degrees of freedom):

$$\begin{aligned}
 & \begin{bmatrix} A_2^{11} & A_2^{13} \\ A_2^{31} & A_2^{33} \end{bmatrix} \ddot{\vec{x}} + \begin{bmatrix} A_1^{11} & A_1^{13} \\ A_1^{31} & A_1^{33} \end{bmatrix} \dot{\vec{x}} \\
 & + \begin{bmatrix} A_0^{11} - A_0^{12} (A_0^{22})^{-1} A_0^{21} & A_0^{13} - A_0^{12} (A_0^{22})^{-1} A_0^{23} \\ A_0^{31} - A_0^{32} (A_0^{22})^{-1} A_0^{21} & A_0^{33} - A_0^{32} (A_0^{22})^{-1} A_0^{23} \end{bmatrix} \vec{x} \\
 & = \begin{bmatrix} B^1 - A_0^{12} (A_0^{22})^{-1} B^2 \\ B^3 - A_0^{32} (A_0^{22})^{-1} B^2 \end{bmatrix} \vec{v} \\
 & + \begin{bmatrix} B_0^1 - A_0^{12} (A_0^{22})^{-1} B_0^2 \\ B_0^3 - A_0^{32} (A_0^{22})^{-1} B_0^2 \end{bmatrix} \vec{z}
 \end{aligned}$$

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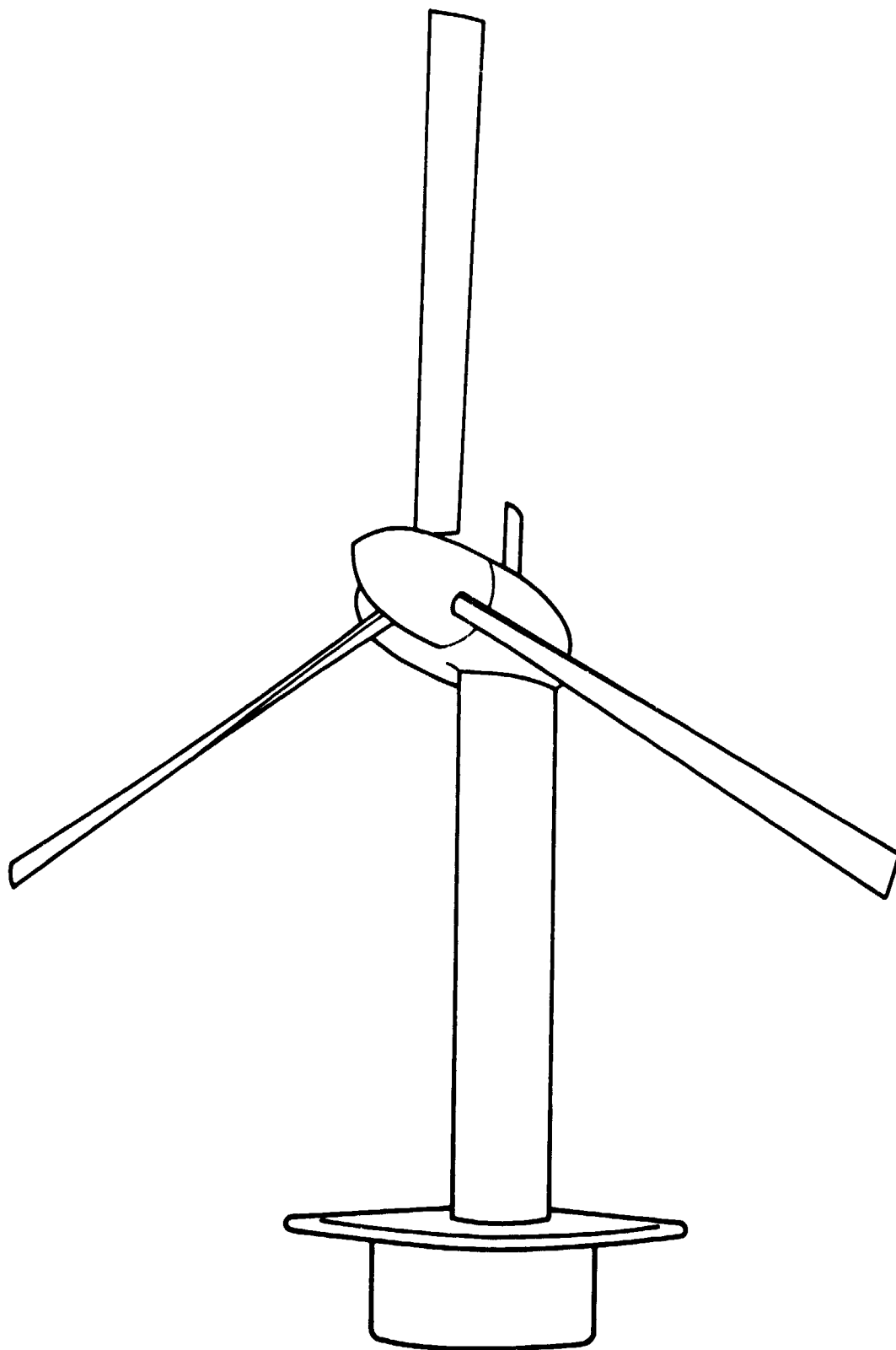


Figure 1. Proprotor and cantilever wing configuration.

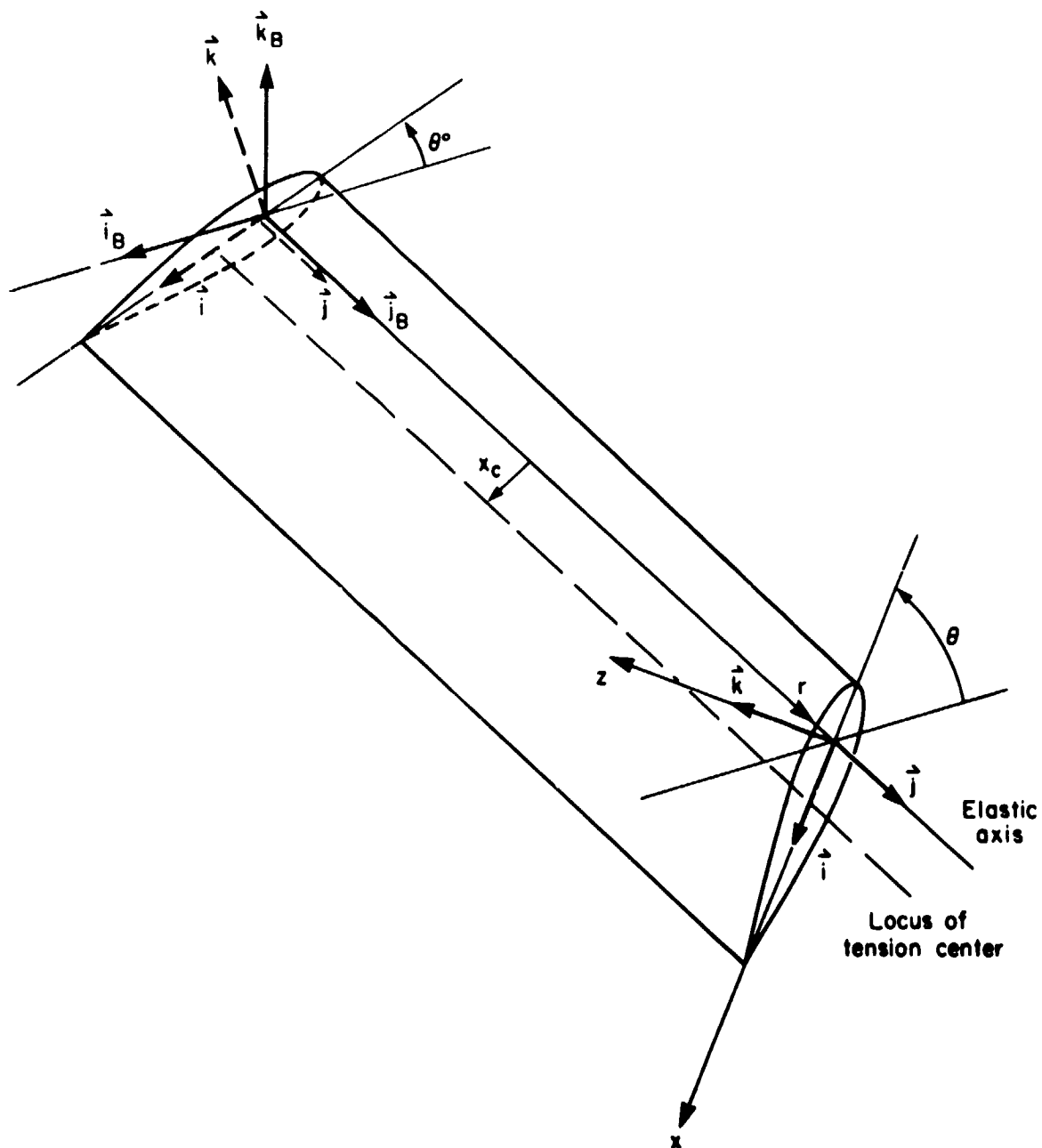


Figure 2. Geometry of undeformed blade.

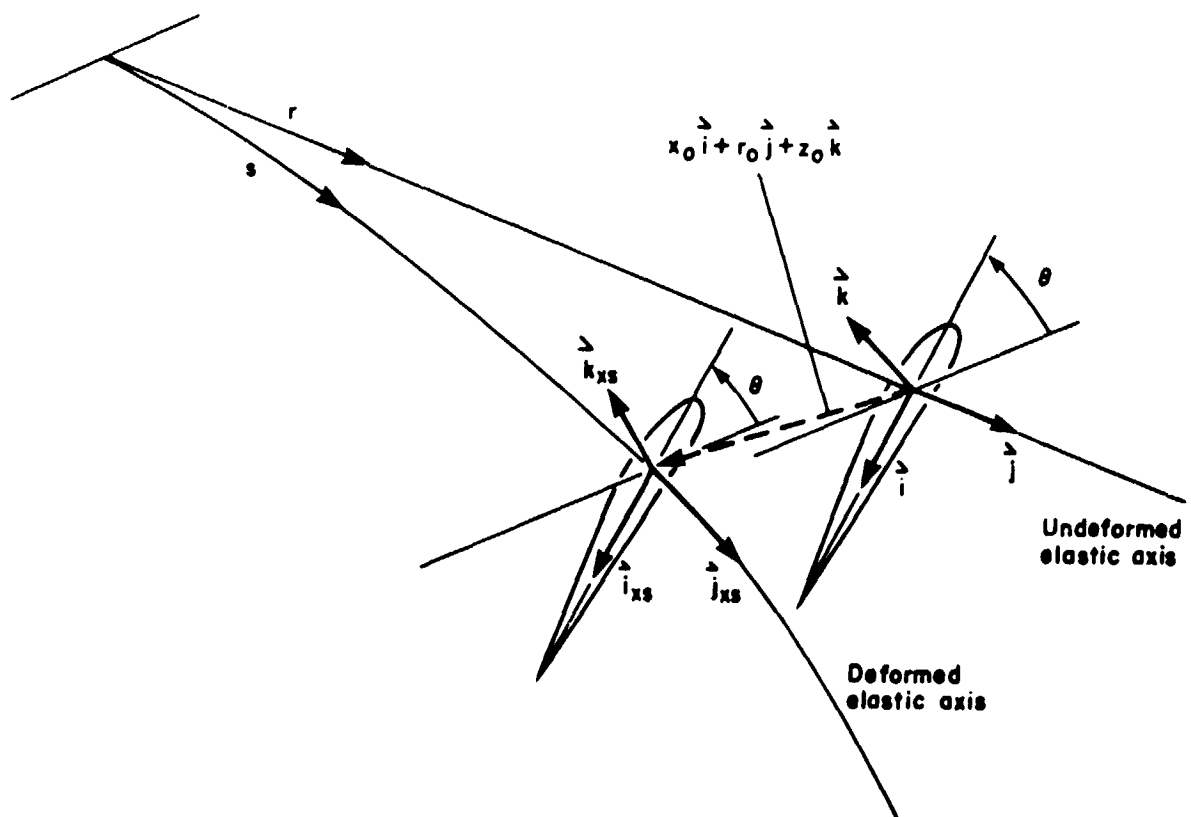


Figure 3. Geometry of deformed blade.

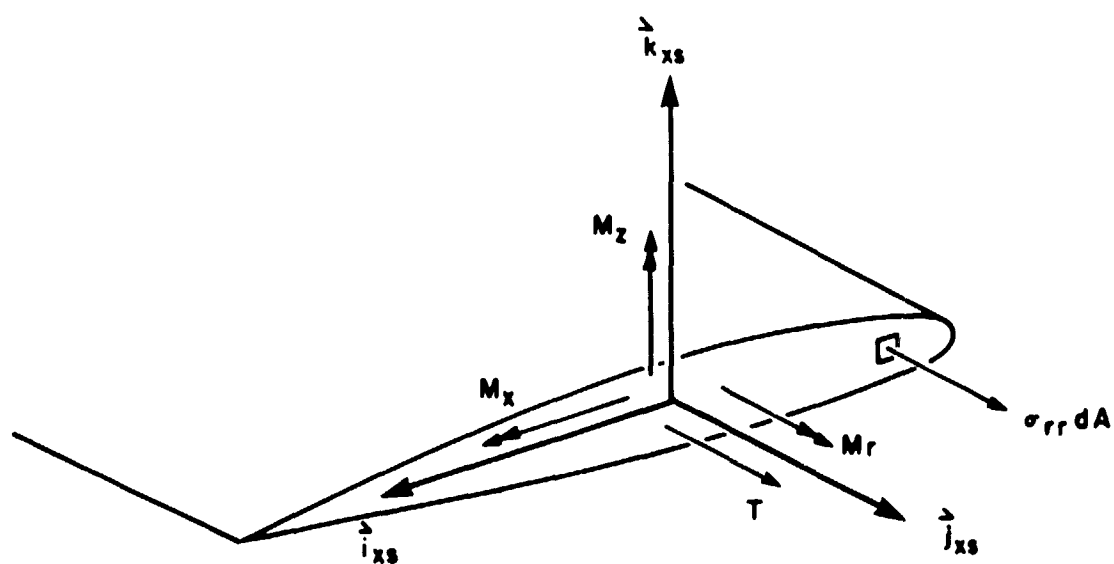


Figure 4. Moments on blade section.

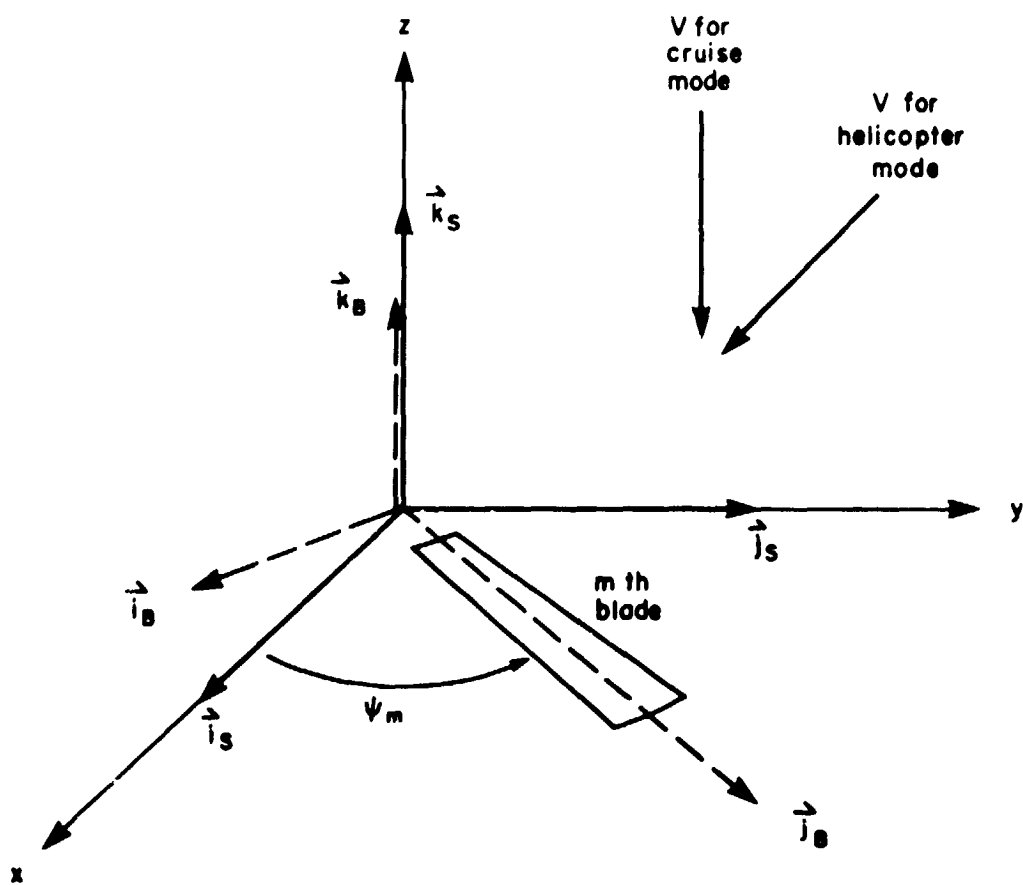


Figure 5. Hub frame coordinate systems (rotating and nonrotating).

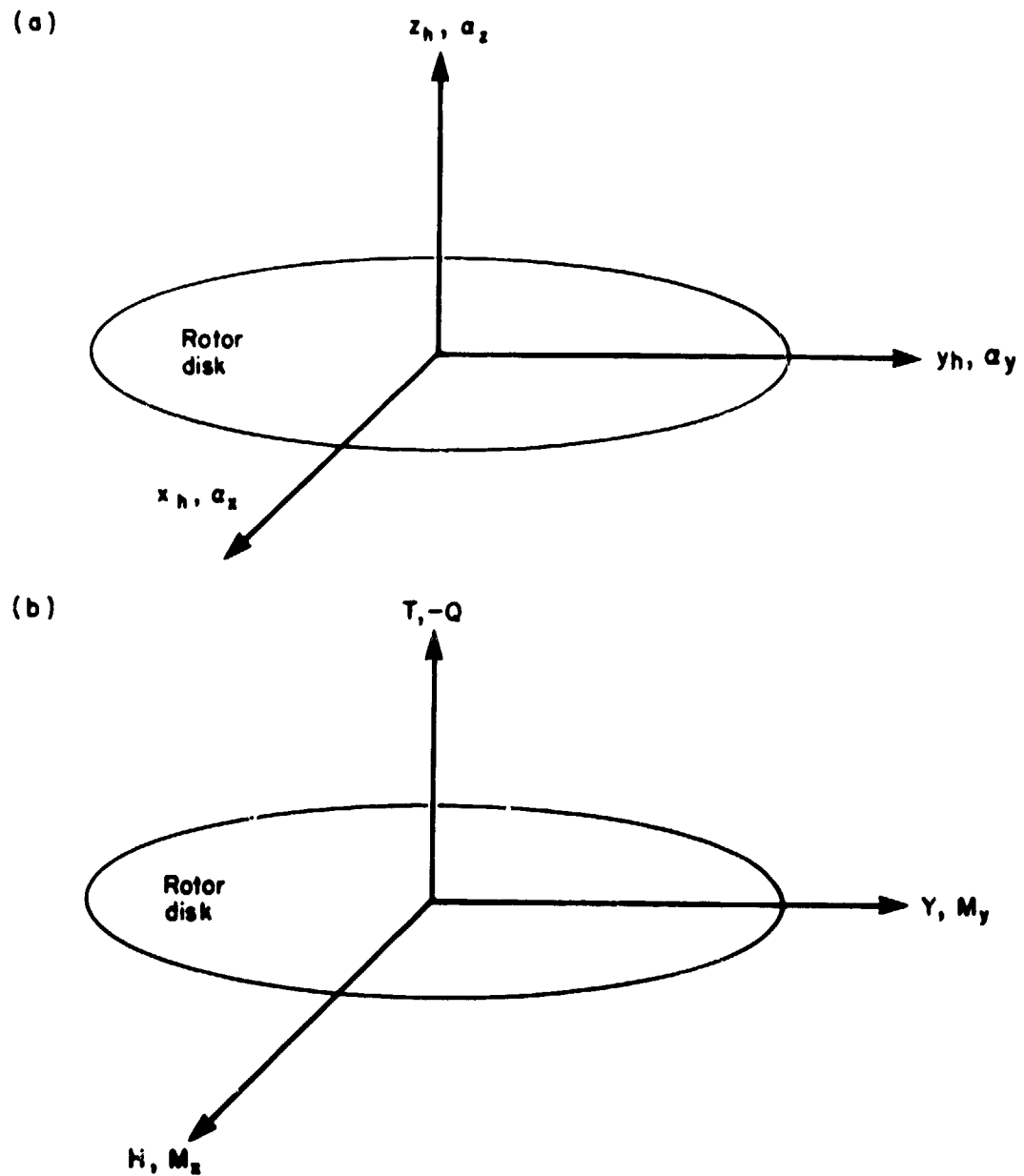


Figure 6. Notation and sign conventions for (a) shaft motion, angular and linear displacement in an inertial frame; and (b) hub forces and moments, on the hub in a nonrotating frame.



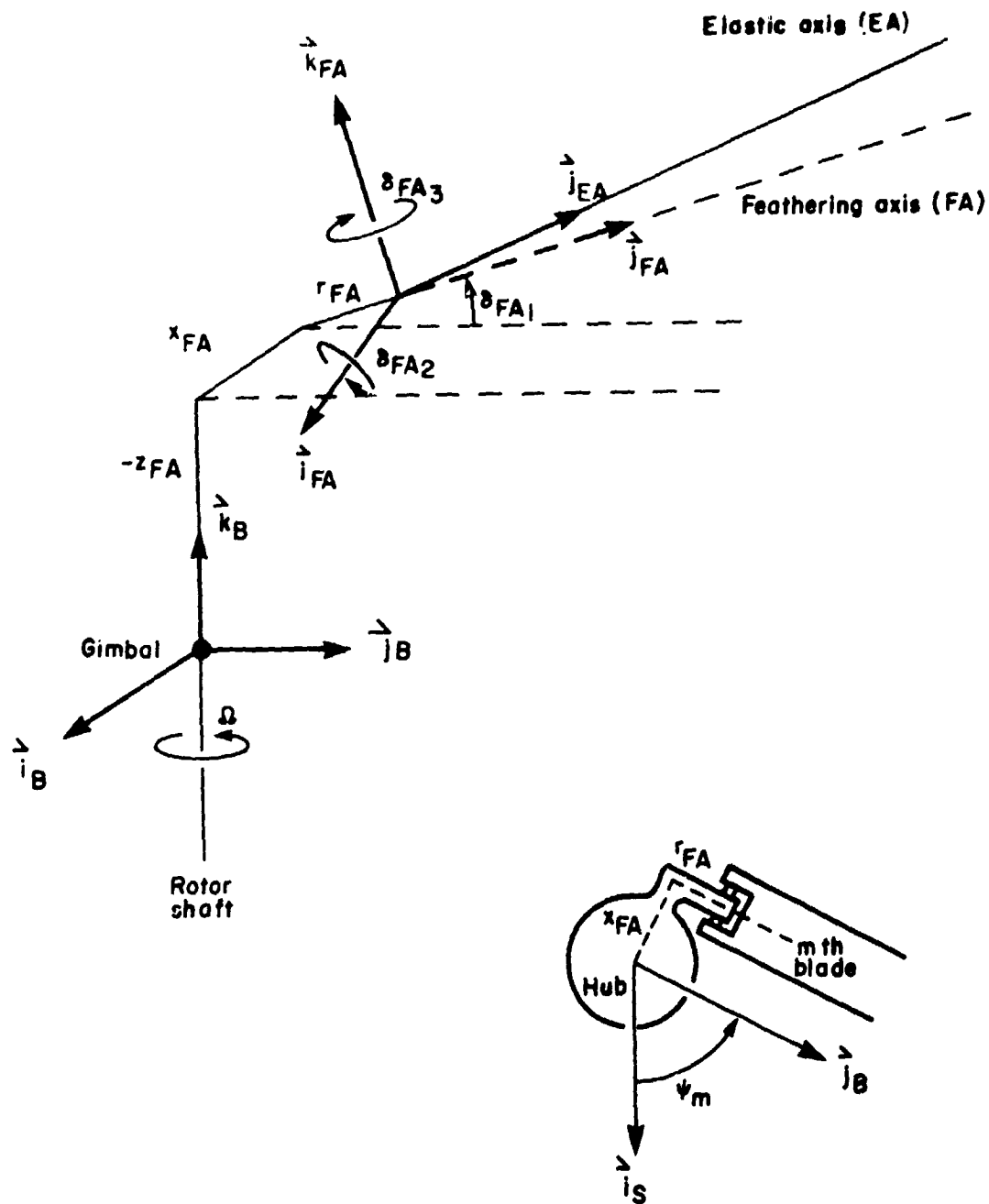


Figure 7. Rotor hub and root geometry (undistorted), showing gimbal undersling ( $z_{FA}$ ), torque offset ( $x_{FA}$ ), feathering axis offset ( $r_{FA}$ ), precone ( $\delta_{FA1}$ ), droop ( $\delta_{FA2}$ ), and sweep ( $\delta_{FA3}$ ).

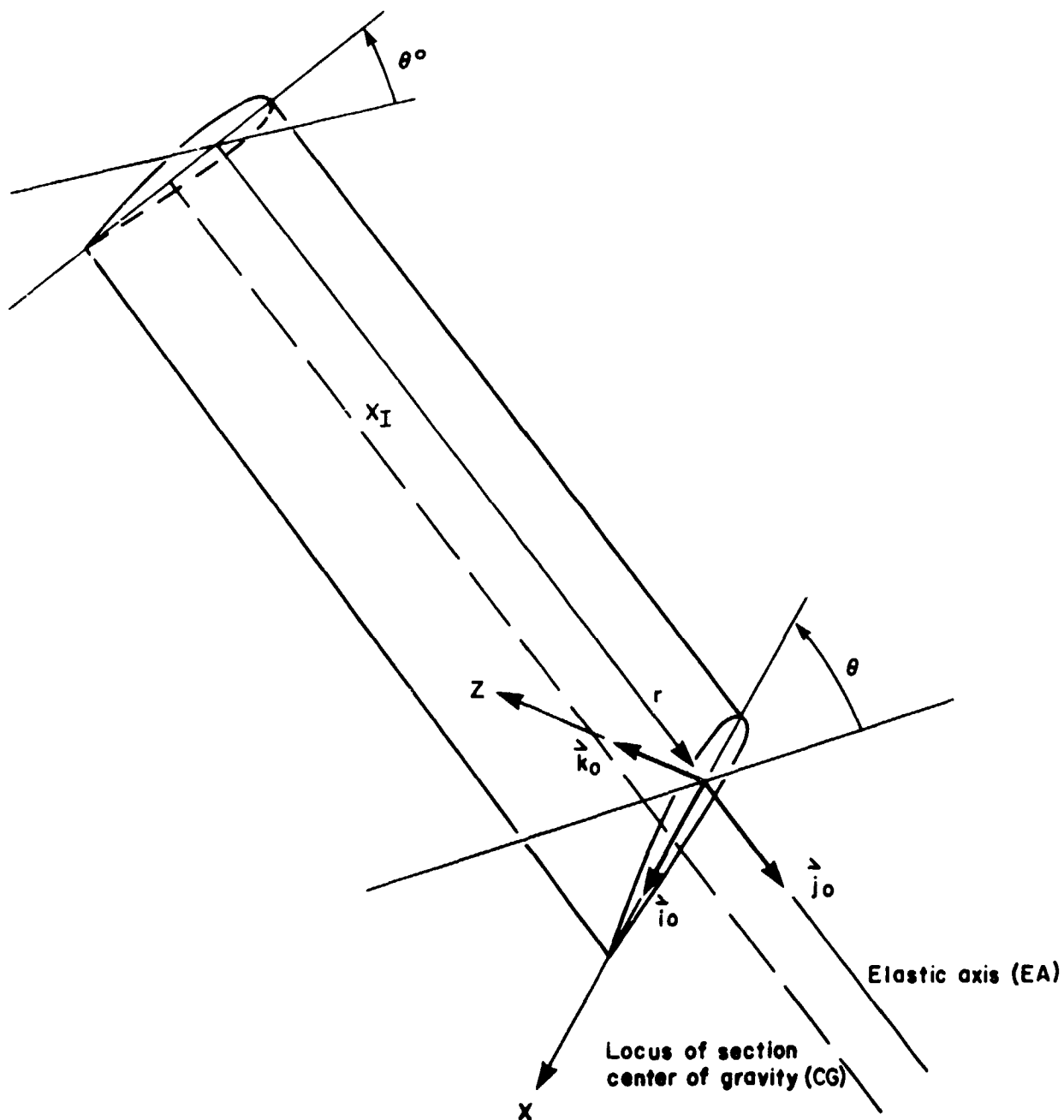
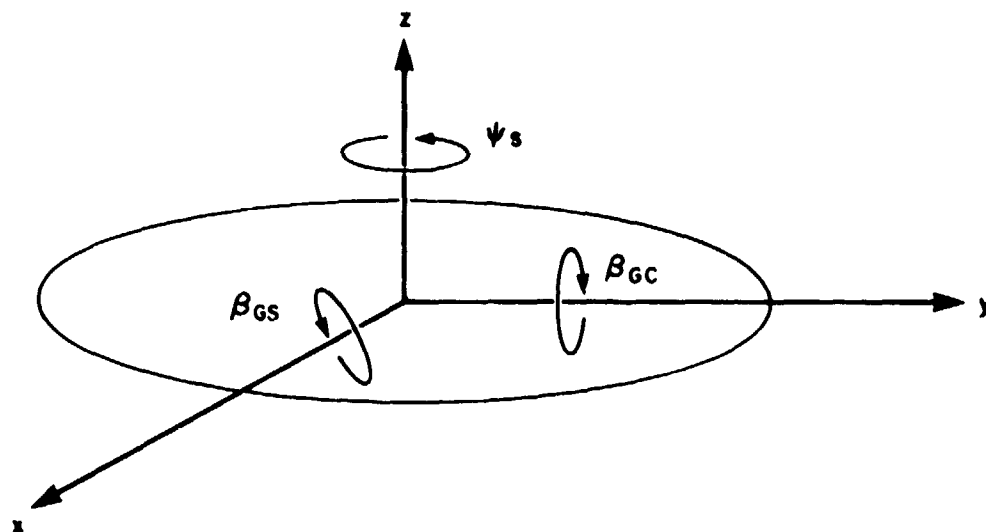


Figure 8. Geometry of undeformed blade.

(a)



(b)

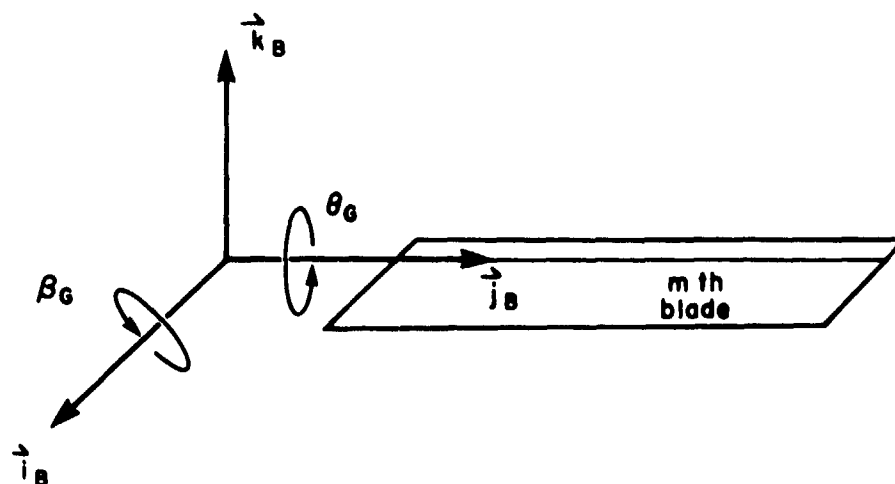


Figure 9. Notation and sign convention for gimbal motion, (a) in the nonrotating frame, and (b) in the rotating frame.

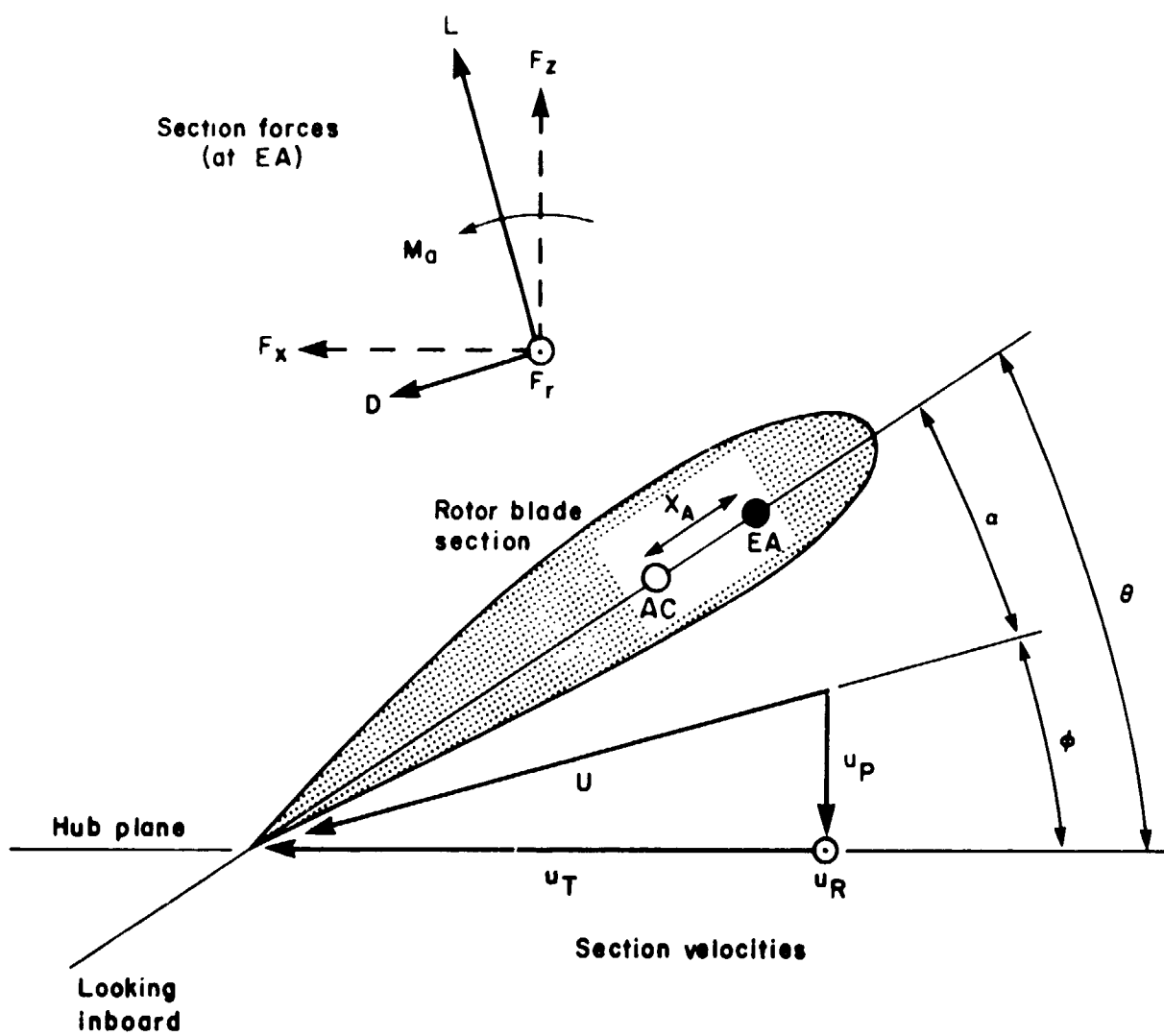


Figure 10. Rotor blade section aerodynamics; notation and sign conventions for section forces and velocities.

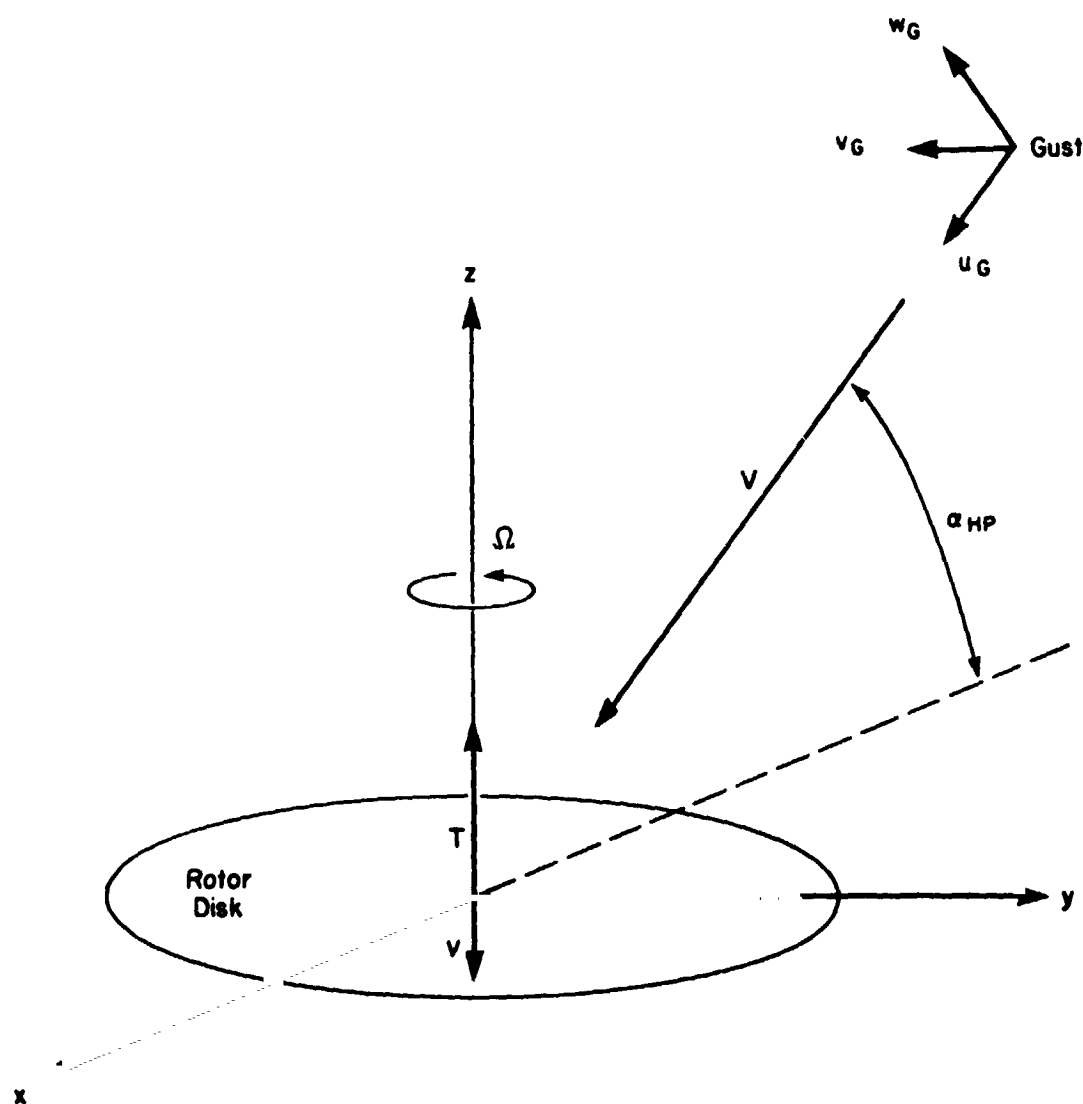


Figure 11. Notation and sign conventions for rotor velocity and orientation ( $V$  and  $\alpha_{HP}$ ), induced velocity ( $v$ ), and aerodynamic gust velocity components ( $u_G$ ,  $v_G$ ,  $w_G$ ).

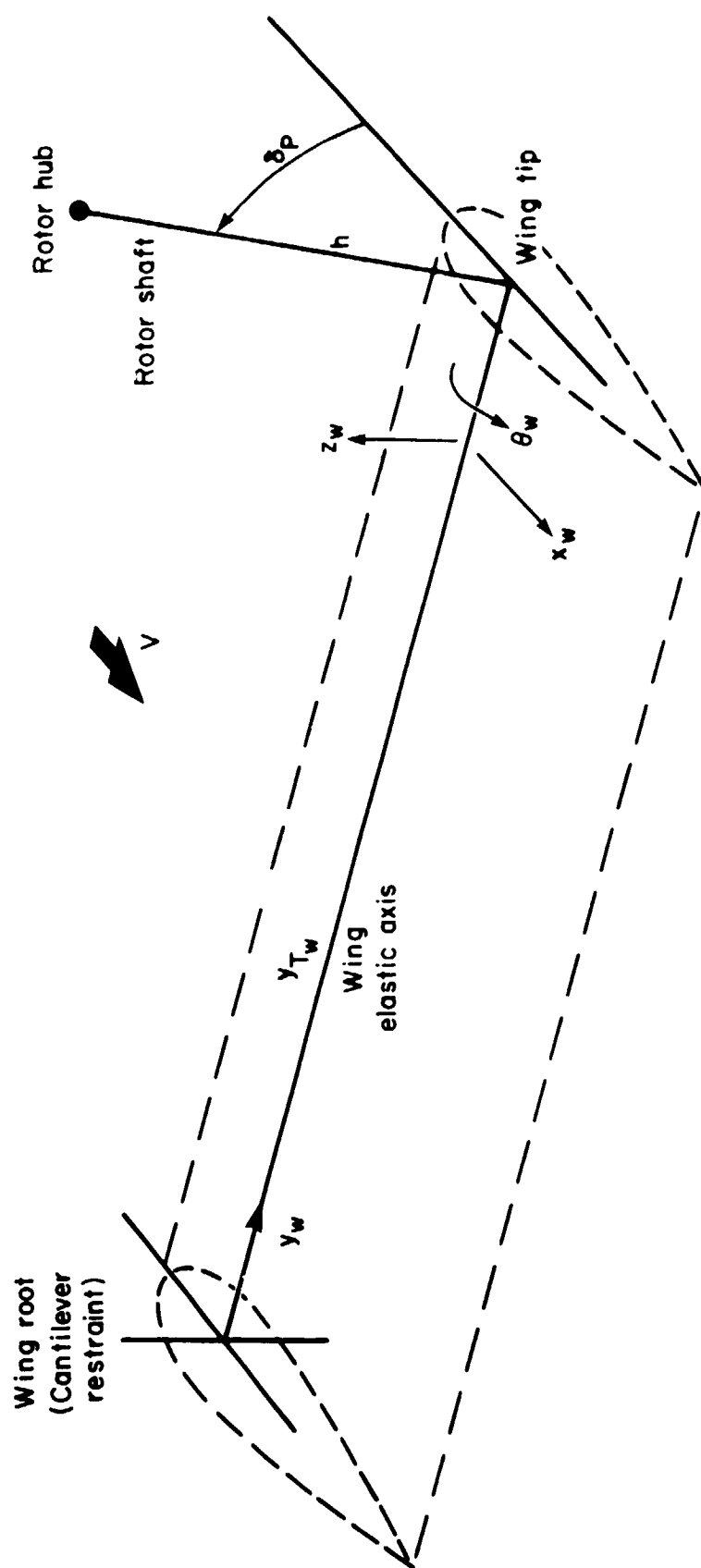


Figure 12. Geometry of cantilever wing and rotor shaft orientation.

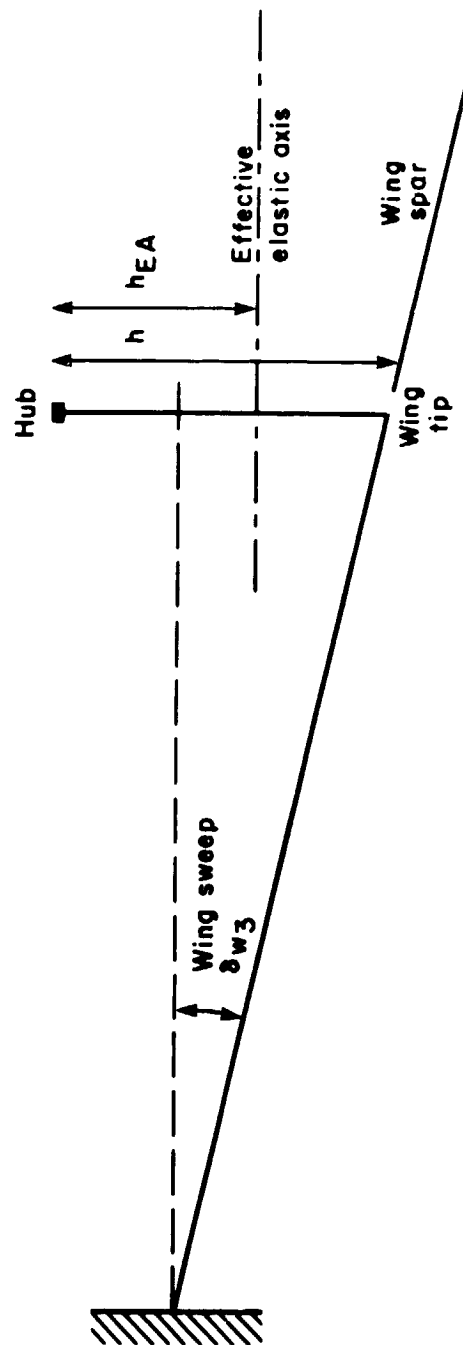


Figure 12. Geometry of wing, effective elastic axis.

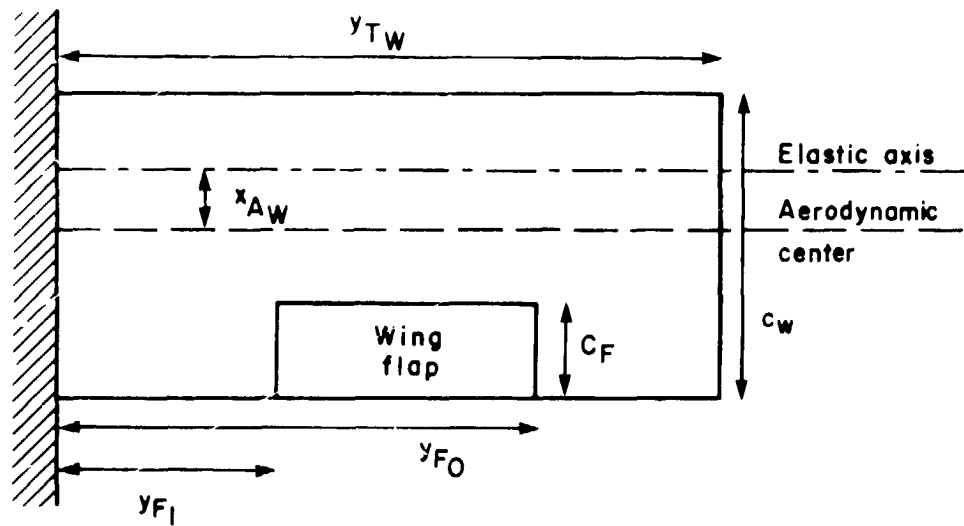


Figure 14. Geometry of wing and wing flap aerodynamics.